Circular Cylinders
Flow past a circular cylinder

1. Flow past a circular cylinder without circulation.
2. Adding circulation.
3. Circular cylinder in non-uniform flow

Why do we care so much about circular cylinders?
1. Acyclic Flow Past a Circular Cylinder

\[ F(z) = V_\infty e^{-i\alpha} z + \frac{V_\infty a^2 e^{i\alpha}}{z - z_1} \]

\[ W(z) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(z - z_1)^2} \]
1. Acyclic Flow Past a Circular Cylinder

Prove it’s circular?

Pressure distribution on its surface?

\[
F(z) = V_\infty e^{-i\alpha}z + \frac{V_\infty a^2 e^{i\alpha}}{z - z_1}
\]

\[
W(z) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(z - z_1)^2}
\]
1. Acyclic Flow Past a Circular Cylinder

Comparison with real life
2. Circular Cylinder with Circulation

= Uniform flow + Opposing Doublet + Vortex

So, for a cylinder radius $a$, centered at $z_1$ in a free stream of velocity $V_\infty$ at angle $\alpha$ to the $x$ axis with circulation $\Gamma$:

$$F(z) = V_\infty e^{-i\alpha} z + \frac{V_\infty a^2 e^{i\alpha}}{z - z_1} - \frac{i\Gamma}{2\pi} \log_e(z - z_1)$$

$$W(z) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(z - z_1)^2} - \frac{i\Gamma}{2\pi(z - z_1)}$$

Velocity on cylinder surface
(take $\alpha = z_1 = 0$ and $z = ae^{i\theta}$)

$$W(z) = V_\infty - \frac{V_\infty a^2}{z^2} - \frac{i\Gamma}{2\pi z}$$

$$v_r - iv_\theta = W(z)e^{i\theta} = V_\infty e^{i\theta} - \frac{V_\infty a^2 e^{i\theta}}{z^2} - \frac{i\Gamma e^{i\theta}}{2\pi z}$$

$$v_\theta = -2V_\infty \sin \theta + \frac{\Gamma}{2\pi a} v_r = 0$$

Pressure and Stagnation Points

$$v_\theta = 0 = -2V_\infty \sin \theta_{stag} + \frac{\Gamma}{2\pi a}$$

$$\theta_{stag} = \arcsin \frac{\Gamma}{4\pi a V_\infty}$$

$$C_p\big|_{surf} = 1 - v_\theta^2 / V_\infty^2$$

$$= 1 - 4 \sin^2 \theta - 4 \left(\frac{\Gamma}{4\pi a V_\infty}\right)^2 + 8 \sin \theta \left(\frac{\Gamma}{4\pi a V_\infty}\right)$$
2. Circular Cylinder with Circulation

\[ \frac{\Gamma}{4\pi a V_{\infty}} \]

\[ \theta_{stag} = \arcsin \frac{\Gamma}{4\pi a V_{\infty}} \]

<table>
<thead>
<tr>
<th>( \frac{\Gamma}{4\pi a V_{\infty}} )</th>
<th>( \theta_{stag} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0°, 180°</td>
</tr>
<tr>
<td>-0.5</td>
<td>-30°, 210°</td>
</tr>
<tr>
<td>-1</td>
<td>-90°, 270°</td>
</tr>
<tr>
<td>-1.5</td>
<td>?</td>
</tr>
</tbody>
</table>
2. Circular Cylinder with Circulation

\[
\frac{\Gamma}{4\pi a V_\infty} = 0
\]

\[
= -0.5
\]

\[
= -1.0
\]

\[
= -1.5
\]

\[
C_p \bigg|_{surf} = 1 - 4\sin^2 \theta - 4 \left( \frac{\Gamma}{4\pi a V_\infty} \right)^2 + 8\sin \theta \left( \frac{\Gamma}{4\pi a V_\infty} \right)
\]
3. Circular Cylinder in Non-Uniform Flow: The Milne-Thompson Circle Theorem

"Consider a flow with complex potential $F(z)$. Let the flow have no rigid boundaries and let all the singularities be greater than a distance 'a' from the origin. If a circular cylinder of radius 'a' is introduced to the flow, centered at the origin, then the complex potential will become: $F_1(z) = F(z) + F(a^2/z)$"
3. Milne-Thompson Theorem: Proof

\[ F_1(z) = F(z) + \overline{F}(a^2/z) \]

\( \overline{F}(z) \) is the conjugate function of \( F(z) \) – the same except with all constants replaced by their conjugates. E.g

\[
F(z) = -\frac{i\Gamma}{2\pi} \log_e (z - z_1)
\]

\[
\overline{F}(z) = \frac{i\Gamma}{2\pi} \log_e (z - \overline{z}_1)
\]

Note that \( \overline{F}(z) = \overline{F}(z) \)
3. Milne-Thompson Theorem: Examples

\[ F_1(z) = F(z) + \overline{F}(a^2/z) \]

(a) Circle in a uniform flow in \( x \) direction:

\[ F(z) = V_\infty z \]

(b) Circle in a source flow, source at \( z_1 \):

\[ F(z) = \frac{q}{2\pi} \log_e (z - z_1) \]
(b) Circle in a source flow, source at $z_1$ (contd.)

“Image of a source in a circle is a source of equal strength at the inverse point and a sink of opposite strength at the circle center”
(Results similar for vortex or doublet)

Convection? Force on the cylinder?