Intro to 2D Ideal Flow
2D Steady Ideal Flow

- Governing equations
  - Continuity \( \nabla \cdot \mathbf{V} = 0 \quad \nabla^2 \phi = 0 \)
  - Bernoulli \( p + \frac{1}{2} \rho V^2 = \text{const.} \)
  - Irrotationality \( \nabla \times \mathbf{V} = 0 \quad \mathbf{V} = \nabla \phi \)

- Other results that matter
  - Circulation must be specified around closed bodies
  - Boundary condition \( \mathbf{V} \cdot \mathbf{n} = 0 \) is same as condition on any streamline
  - Kutta-Joukowksi theorem (force on a body in undisturbed fluid)

\[
\mathbf{F} = -\rho \mathbf{U}_b \times \left( \begin{array}{c}
\mathbf{i} \int \Gamma_x \, dx + \mathbf{j} \int \Gamma_y \, dy + \mathbf{k} \int \Gamma_z \, dz
\end{array} \right)_{X \text{ body}}_{Y \text{ body}}_{Z \text{ body}}
\]
Relations for flow in the x, y plane

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<td>Relationships</td>
<td>[ \phi(B) - \phi(A) = \int_A^B \mathbf{V} \cdot \mathbf{e}_s , ds ]</td>
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Complex Numbers

- A complex number $z$ can ALWAYS be separated into its real $x$ and imaginary $iy$ parts and that separation is UNIQUE.

- Magnitude $= |z| = \sqrt{x^2 + y^2}$

- Argument (or angle) $= \arg(z) = \arctan(y/x)$

- A single complex variable can be used to represent real independent variables with the same dimensions. E.g. Coordinates, velocities, streamfunction/potential.

- Any complex variable may be expressed in polar form using the complex exponential. E.g. Coordinates $z = x + iy = r \cos \theta + ir \sin \theta = re^{i\theta}$.

- Multiplying two complex numbers/variables together multiplies their magnitudes and adds their angles.

- Multiplication of any complex number by $e^{i\theta}$ is equivalent to rotation about the origin through angle $\theta$.

- Raising a complex number to a power multiplies its argument by that power.

Position $z = x + iy$
Complex Velocity $W(z) = u - iv$
Complex Potential $F(z) = \phi + i \psi$
Complex Functions I

• A complex function [e.g. $F(z)$, $W(z)$, $\sin(z)$, $\zeta(z)$] produces a complex number at every position $z$. That number can always be split into real and imaginary parts, say $\zeta(z) = \xi(x,y) + i\eta(x,y)$ where $\xi$ and $\eta$ are real functions.

• Now consider differentiating the function (w.r.t. $z$), e.g.
• The derivatives contain no reference to the direction of $z$ so

\[
\frac{d\zeta}{dz} = \frac{\partial \zeta}{\partial x} = \frac{\partial \zeta}{\partial (iy)} = -i \frac{\partial \zeta}{\partial y}
\]

\[
\frac{\partial \zeta}{\partial x} = -i \frac{\partial \zeta}{\partial y}
\]

\[
\frac{\partial \zeta}{\partial x} = \frac{\partial \eta}{\partial y}, \quad \frac{\partial \zeta}{\partial y} = -\frac{\partial \eta}{\partial x}
\]

\[
\frac{\partial \zeta}{\partial x} + i \frac{\partial \eta}{\partial x} = -i \frac{\partial \zeta}{\partial y} - i.i \frac{\partial \eta}{\partial y}
\]

• This result is called the CAUCHY RIEMANN CONDITIONS
• Differentiating these conditions (w.r.t. $x$ or $y$ gives)

\[
\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} = 0, \quad \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0
\]

• The real and imaginary parts of any differentiable complex function of $z$ are solutions to Laplace's equation

ANALYTIC FUNCTION
2D Ideal Flow in the Complex Plane

Summary

• Position: $z = x + iy$

• Complex velocity: $W(z) = u - iv$

• Complex potential: $F(z) = \phi + i\psi$

• As analytic functions $W(z)$ and $F(z)$ automatically satisfy the governing equations and the relations between dependent variables.

• Also note: $W = dF/dz$  \hspace{2cm} F = \int Wdz$

  \[ \Gamma + iQ = \oint Wdz \hspace{2cm} \nu_r - i\nu_\theta = (u - iv)e^{i\theta} = We^{i\theta} \]