MS66: Topology and Mixing in Fluids

- **Mark Stremler**, Virginia Tech *(replacing P. Boyland)*
  Topological chaos in cavities and channels

- **Matt Finn**, University of Adelaide
  Topological entropy of braids on the torus

- **Tsuyoshi Kobayashi**, Nara Women’s U., Japan
  Realizing topological chaos with simple mechanisms

- **Kai de Lange Kristiansen**, UCSB
  Braid theory and microparticle dynamics in ferrofluids
Topological chaos and fluid mixing in cavities and channels

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Topological chaos

Complexity is ‘built in’ the flow due to the topology of the boundary motions

\( R_N \): 2D fluid region with \( N \) stirring ‘rods’

- stirrers move on periodic orbits
- stirrers = solid objects or fluid particles


Topological chaos

Complexity is ‘built in’ the flow due to the topology of the boundary motions

\[ R_N : \text{2D fluid region with } N \text{ stirring ‘rods’} \]

- stirrers move on periodic orbits
- stirrers = solid objects or fluid particles
- stirrer motions generate a mapping
  \[ f : R_N \rightarrow R_N \quad \text{(a diffeomorphism)} \]
- stirrer trajectories generate braids in 2+1 dimensional space-time
Thurston-Nielsen Theory

- Casson & Bleiler (1988) *Automorphisms...*

A stirrer motion $f$ is isotopic to a stirrer motion $g$ of one of three types:
Thurston-Nielsen Theory

A stirrer motion $f$ is isotopic to a stirrer motion $g$ of one of three types:

(i) finite order (f.o.): the $n$th iterate of $g$ is the identity

(ii) pseudo-Anosov (pA): $g$ has dense orbits, Markov partition with transition matrix $A$

$\lambda > 1 :$ expansion or dilation $= \text{PF eigenvalue of } A$

entropy equal to the topological entropy $h_{\text{top}}(g) = \log(\lambda)$

(iii) reducible: $g$ contains both f.o. and pA regions
Isotopy

• Fixed stirrer motions:
  all fluid motions with stirrers fixed in space, with rotations allowed

• Isotopy to the identity:
  a stirrer motion $h$ is isotopic to the identity if the same result could have been obtained by a fixed stirrer motion
  – for 0 or 1 stirrer, all motions are isotopic to the identity
  – for 2 stirrers, all motions (or 2nd iterate) are isotopic to the identity

• Isotopic motions:
  motions $f$ and $g$ are isotopic if $g = hf$, with $h$ isotopic to the identity

• Handel’s isotopy stability theorem:
  the complex dynamics of the pA map remains under isotopy
  $\log(\lambda)$ provides a lower bound on the topological entropy
**Braids on 3 strands or Stirring with 3 rods**

**Finite Order:** $\sigma_1 \sigma_2$

fix the rods and ‘unwind’ the outer boundary
no lower bound for stretching in the flow

**pseudo Anosov:** $\sigma_1^{-1} \sigma_2$

complexity cannot be removed with rods fixed

$$\lambda = \frac{1}{2} (3 + \sqrt{5})$$

every topologically non-trivial curve grows in length like $\lambda^n$ under iteration
Stirring experiments with 3 rods

finite order

\[ R_+ \]

\[ L_+ \]

pseudo Anosov
Stirring with ‘ghost rods’


Finn, Thiffeault & Gouillart (2006) Physica D

Topological chaos in doubly-periodic sine flow with no rods

Can we generate topological chaos in a ‘realistic’ flow without using any solid stirring rods?
Lid-driven cavity flow


steady or time dependent 2D flow in a rectangular cavity

steady forcing

periodic forcing

Initial

Final
Lid-driven cavity flow


steady or time dependent 2D flow in a rectangular cavity

steady forcing

periodic forcing
Our lid-driven cavity flow

top boundary split into 3 segments
other boundaries are fixed

time-periodic operation of steady Stokes flow

design parameters: boundary velocities
aspect ratio $\alpha = a/b$
boundary length ratio $\beta = 2c/d$

find periodic points that act as stirring rods
Our lid-driven cavity flow

Numerical solution of the 2D biharmonic equation for the streamfunction in Stokes flow


$$\nabla^2 \nabla^2 \psi(x, y) = 0$$

$$x = \pm a : \quad \psi = 0, \quad \partial \psi / \partial x = 0$$

$$y = b : \quad \psi = 0, \quad \partial \psi / \partial y = V(x)$$

$$y = -b : \quad \psi = 0, \quad \partial \psi / \partial y = 0$$

$$\psi(x, y) = \psi_{ee}(x, y) + \psi_{eo}(x, y) + \psi_{oe}(x, y) + \psi_{oo}(x, y)$$
Our lid-driven cavity flow

\[ \psi_{ee}(x, y) = b \sum_{m=1}^{\infty} \frac{(-1)^m}{\alpha_m} P_m p_m(y) \cos(\alpha_m x) - a \sum_{n=1}^{\infty} \frac{(-1)^n}{\beta_n} Q_n q_n(x) \cos(\beta_n y) \]

\[ x = \pm a : \quad \psi_{ee} = 0, \quad \frac{\partial \psi_{ee}}{\partial x} = 0 \]

\[ y = \pm b : \quad \psi_{ee} = 0, \quad \frac{\partial \psi_{ee}}{\partial y} = \pm U_{ee}(x) \]

\[ p_m(y) = b \tanh(\alpha_m b) \frac{\cosh(\alpha_m y)}{\cosh(\alpha_m b)} - y \frac{\sinh(\alpha_m y)}{\cosh(\alpha_m b)} \]

\[ \alpha_m = \frac{(2m - 1)\pi}{2a} \]

\[ q_n(x) = a \tanh(\beta_n a) \frac{\cosh(\beta_n x)}{\cosh(\beta_n a)} - x \frac{\sinh(\beta_n x)}{\cosh(\beta_n a)} \]

\[ \beta_n = \frac{(2n - 1)\pi}{2b} \]
Our lid-driven cavity flow

\[ \psi_{ee}(x, y) = b \sum_{m=1}^{\infty} \frac{(-1)^m}{\alpha_m} P_m p_m(y) \cos(\alpha_m x) - a \sum_{n=1}^{\infty} \frac{(-1)^n}{\beta_n} Q_n q_n(x) \cos(\beta_n y) \]

key to the solution: assume

\[ P_m = P_0 + \xi_m \]
\[ Q_n = Q_0 + \eta_n \]

infinite system of equations:

\[ \xi_m b \triangle(\alpha_m b) = \sum_{n=1}^{\infty} \eta_n \frac{4\alpha_m^2 \beta_n}{(\alpha_m^2 + \beta_n^2)^2} + F_m(P_0, Q_0, \alpha_m, \beta_n, U_m) \]
\[ \eta_n a \triangle(\beta_n a) = \sum_{m=1}^{\infty} \xi_m \frac{4\beta_n^2 \alpha_m}{(\alpha_m^2 + \beta_n^2)^2} + H_n(P_0, Q_0, \alpha_m, \beta_n) \]
Our lid-driven cavity flow

asymptotic approximation for $P_0, Q_0$

solve finite system for $\xi_m, \eta_n$

\[
\psi_{ee}(x, y) = b \sum_{m=1}^{\infty} \frac{(-1)^m}{\alpha_m} P_m p_m(y) \cos(\alpha_m x) - a \sum_{n=1}^{\infty} \frac{(-1)^n}{\beta_n} Q_n q_n(x) \cos(\beta_n y)
\]

\[
= b \sum_{m=1}^{M} \frac{(-1)^m}{\alpha_m} \xi_m p_m(y) \cos(\alpha_m x) - a \sum_{n=1}^{N} \frac{(-1)^n}{\beta_n} \eta_n q_n(x) \cos(\beta_n y)
\]

\[
+ P_0 b \sum_{m=1}^{M+M'} \frac{(-1)^m}{\alpha_m} p_m(y) \cos(\alpha_m x) - Q_0 a \sum_{n=1}^{N+N'} \frac{(-1)^n}{\beta_n} q_n(x) \cos(\beta_n y)
\]
Topological chaos in lid-driven cavity flow

\[ \alpha = \frac{a}{b} = 3 \quad \beta = \frac{2c}{d} = 1 \]

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\[ \alpha = \frac{a}{b} = 3 \]

\[ x_C = (0, y_0) \]

\[ x_R = (x_1, y_1) \]

change position after time \( \tau \approx 5.379 \)

\[ x_L = (-x_1, y_1) \]

is a stagnation point

puncture the domain at the periodic orbits, examine the motion
Topological chaos in lid-driven cavity flow

Poincaré section
‘design points’ are hyperbolic

elliptic point trajectories
elliptic orbits give a braid on 6 strands

motion is reducible to a pA braid on 3 strands

\[ h_{\text{top}} = \log \left[ \frac{1}{2} (3 + \sqrt{5}) \right] \approx 0.96 \]
Stretching of non-trivial material lines

\[ h_{\text{flow}} \approx 1.92 \approx 2 h_{\text{top}} \]

Finn, Cox & Byrne (2003) JFM

\[ h_{\text{flow}} \approx 0.80 < h_{\text{top}} \]
other cases

aspect ratio \( \alpha = a/b \)

boundary length ratio \( \beta = 2c/d \)

how does this complexity hold up under perturbation?
Extension to three dimensions

Braided Pipe Mixer
Finn, Cox & Byrne (2003) *Phys. Fluids*

braided pipe inserts do not mix well
‘Lid-driven’ channel flow

steady 3D flow in a rectangular channel

- surface grooves – Stroock et al. (2002) *Science*

*lid-driven cavity flow + channel flow*

mixes well
Topological chaos in a ‘lid-driven’ channel

\[ \begin{align*}
\alpha &= 2 \\
\beta &= 2 \\
V_{\text{max}} &= U \\
l &\approx 5.372b
\end{align*} \]

lid-driven secondary flow + axial Poiseuille flow (V)
Topological chaos in a ‘lid-driven’ channel

- How does this compare with the Braided Pipe Mixer?
- How does the stretching compare with the 2D cases?
- Does the braiding matter in this flow?
- How is this affected by perturbations?