On the statics and dynamics of point vortices in periodic domains

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funded by the U.S. National Science Foundation
point vortex systems with $N$ ‘base’ vortices

vortex strengths: $\Gamma_n$

vortex positions: $z_n = x_n + i y_n$

singly periodic domain or periodic strip

domain period: $L$

image positions: $z_n + k L$

$\forall k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
point vortex systems with $N$ ‘base’ vortices

vortex strengths: $\Gamma_n$

vortex positions:

$$z_n = x_n + i y_n$$

doubly periodic domain or periodic parallelogram

domain half-periods: $\omega_1, \omega_2$

domain area: $\Delta$

image positions:

$$z_n + 2j \omega_1 + 2k \omega_2$$

$\forall j, k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
Motivation for considering periodic vortex systems includes...

- vortex wakes behind oscillating bluff bodies

Wake of an oscillating cylinder from C.H.K. Williamson

Motivation for considering periodic vortex systems includes...

- self-organization of vortices in 2D turbulence

McWilliams (1990) *JFM*: dynamics of individual vortices

depends on initial conditions

equation

data

van Bokhoven, Trieling, Clercx & van Heijst (2007) *Physics of Fluids* 19, 046601

numerical experiments

development depends on initial conditions
Equations of motion... in the unbounded plane

velocity at position $z$ in the complex plane

$$\frac{dz^*}{dt} = \frac{1}{2\pi i} \sum_{n=1}^{N} \frac{\Gamma_n}{z - z_n}$$

velocity of vortex at $z_m$

$$\frac{dz_m^*}{dt} = \lim_{z \to z_m} \left\{ \frac{dz^*}{dt} - \frac{\Gamma_m}{2\pi i} \frac{1}{z - z_m} \right\} = \frac{1}{2\pi i} \sum_{n=1}^{N'} \frac{\Gamma_n}{z_m - z_n}$$

system is Hamiltonian:

$$\Gamma_n \frac{dx_n}{dt} = \frac{\partial H}{\partial y_n} \quad \Gamma_n \frac{dy_n}{dt} = -\frac{\partial H}{\partial x_n}$$

and preserves the linear impulse:

$$\Xi = \sum_{n=1}^{N} \Gamma_n z_n$$
Equations of motion... in the periodic strip

velocity is due to an infinite number of vortices

array of simple poles with residue $\frac{\Gamma_n}{2\pi i}$

\[
\frac{dz^*_m}{dt} = \frac{1}{2\pi i} \sum_{n=1}^{N} \Gamma_n \sum_{k=-\infty}^{\infty} \frac{1}{z_m - (z_n + kL)}
\]

\[
= \frac{1}{2L} \sum_{n=1}^{N} \Gamma_n \cot \left[ \frac{\pi}{L}(z_m - z_n) \right]
\]

system is Hamiltonian and preserves the linear impulse

Friedmann & Poloubarinova (1928)
Equations of motion... in the periodic parallelogram

lattice of simple poles with residue $\Gamma_n/2\pi i$

the Weierstrass zeta function

$\zeta(z; \omega_1, \omega_2)$

$\zeta(z; \omega_1, \omega_2)$ is quasi-periodic: $\zeta(z + 2a\omega_1 + 2b\omega_2) = 2a\eta_1 + 2b\eta_2$

$\eta_1 = \zeta(\omega_1)$ \hspace{10pt} $\eta_2 = \zeta(\omega_2)$

$N=1$

Tkachenko’s simple lattice (1966): $-i\nu^*(z) = \zeta_0(z) = \zeta(z) + \alpha z$

the simple lattice rotates with an angular velocity $\Omega = \Gamma / 2\Delta$
O’Neil (1989): \( N \) point vortex lattices

\[
\frac{dz_m^*}{dt} = \frac{1}{2\pi i} \sum_{n=1}^{N} \Gamma_n \left[ \zeta(z_m - z_n) + \alpha(z_m - z_n) - \frac{\pi}{\Delta} (z_m^* - z_n^*) \right]
\]

limit of an infinite number of vortices – ordered sum

Stremler & Aref (1999):

\[
\alpha = \frac{\pi \omega_1^*}{\omega_1 \Delta} - \frac{\eta_1}{\omega_1}
\]

if \( \sum_{n=1}^{N} \Gamma_n = 0 \):

\[
\frac{-1}{2\pi i} \sum_{n=1}^{N} \Gamma_n \left[ \frac{\pi}{\Delta} (z_m^* - z_n^*) \right] = \frac{-i}{2\Delta} \sum_{n=1}^{N} \Gamma_n z_n^* = \frac{-i \Xi^*}{2\Delta}
\]

constant translation

if \( S = \sum_{n=1}^{N} \Gamma_n \neq 0 \):

\[
\frac{-1}{2\pi i} \sum_{n=1}^{N} \Gamma_n \left[ \frac{\pi}{\Delta} (z_m^* - z_n^*) \right] = \frac{iS}{2\Delta} \left\{ \frac{z_m^* - \Xi^*}{S} \right\}
\]

constant rotation
‘General’ equations of motion

\[ \frac{d z_m^*}{dt} = \frac{1}{2\pi i} \sum_{n=1}^{N} \Gamma_n \Phi_i(z_m - z_n) \]

where in singly periodic domain

\[ \Phi_1(z) = \frac{\pi}{L} \cot \left( \frac{\pi z}{L} \right) \]

and in doubly periodic domain

\[ \Phi_2(z) = \zeta(z; \omega_1, \omega_2) + \left[ \frac{\pi \omega_1^*}{\omega_1 \Delta} - \frac{\eta_1}{\omega_1} \right] z - \frac{\pi}{\Delta} z^* \]

\( \Phi_i(z) \) is odd in \( z \)
$N=2$ vortices in a periodic domain gives relative equilibria...

if $S = \sum_{n=1}^{N} \Gamma_n = 0$:

$$\frac{d\gamma_1^*}{dt} = \frac{d\gamma_2^*}{dt} = \frac{\Gamma}{2\pi i} \Phi_i(z_2 - z_1) = V^*$$

every such 2-vortex configuration is a relative equilibrium
fluid is ‘stretched and folded’ by steady oblique wakes

\[ a \approx 0.455 L \quad \sinh(b \pi / L) = \sin(a \pi / L) \]

\[ i = 1 \]
case: $i = 4$

$t = 120$

$t = 190$

$t = 260$
$N=2$ vortices in a periodic domain gives relative equilibria...

if \( S = \sum_{n=1}^{N} \Gamma_n = 0 \):
\[
\frac{dz_1^*}{dt} = \frac{dz_2^*}{dt} = \frac{\Gamma}{2\pi i} \Phi_i(z_2 - z_1) = V^*
\]

every such 2-vortex configuration is a relative equilibrium

...and \( N=2 \) in a periodic domain gives simple dynamics

if \( S = \sum_{n=1}^{N} \Gamma_n \neq 0 \):
\[
\text{define} \quad Z = z_1 - z_2
\]
\[
\frac{dZ^*}{dt} = \frac{dz_1^*}{dt} - \frac{dz_2^*}{dt} = \frac{S}{2\pi i} \Phi_i(Z)
\]

relative vortex motion reduced to advection of a passive particle by a single (periodic) vortex of strength \( S \) at the origin
advection problem for $Z$ in a periodic strip

\[ \Gamma_1 = 1 \]
\[ \Gamma_2 = 2 \]

corresponding vortex motion in the $z$-plane

\[ \Gamma_1 = -1 \]
\[ \Gamma_2 = 2 \]
advection problem for \( Z \) in a periodic parallelogram

\[
S = \sum_{n=1}^{N} \Gamma_n \neq 0
\]

the doubly-periodic system produces both stable and unstable equilibria

the triangular lattice

Tkachenko (1966) *PLA*; Campbell, Doria & Kadtke (1989) *PRA*
$N=3$ vortices in a periodic domain gives richer dynamics

\[ 2\pi i \frac{dz_1^*}{dt} = \Gamma_2 \Phi_i(z_1 - z_2) - \Gamma_3 \Phi_i(z_3 - z_1) \]

\[ 2\pi i \frac{dz_2^*}{dt} = \Gamma_3 \Phi_i(z_2 - z_3) - \Gamma_1 \Phi_i(z_1 - z_2) \]

\[ 2\pi i \frac{dz_3^*}{dt} = \Gamma_1 \Phi_i(z_3 - z_1) - \Gamma_2 \Phi_i(z_2 - z_3) \]

assume \( S = \sum_{n=1}^{N} \Gamma_n = 0 \)

define \( Z = z_1 - z_2 \)

\[ 2\pi i \frac{dZ^*}{dt} = -\Gamma_3 [\Phi_i(z_1 - z_2) + \Phi_i(z_2 - z_3) + \Phi_i(z_3 - z_1)] \]

use \( S = 0 \) and \( \Xi = \Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 \)

\[ z_2 - z_3 = (\Gamma_1 Z - \Xi) / \Gamma_3 \]

\[ z_3 - z_1 = (\Gamma_2 Z + \Xi) / \Gamma_3 \]
\[ 2\pi i \frac{dZ^*}{dt} = -\Gamma_3 \left\{ \Phi_i(Z) + \Phi_i \left[ (\Gamma_1 Z - \Xi) / \Gamma_3 \right] + \left[ (\Gamma_2 Z + \Xi) / \Gamma_3 \right] \right\} \]

gives an advection problem for \( Z \) in both singly and doubly periodic domains when interpreted properly


strength factors on \( Z \) require advection to occur in a \textit{larger domain}

if \( \Gamma_i \) are rational, then the advection system is \textit{periodic}

if \( \Gamma_i \) are irrational, then the advection system is \textit{aperiodic}

location of the advecting vortices depends on \( \Xi \)
\((\Gamma_1, \Gamma_2, \Gamma_3) = (2, 1, -3)\) in a periodic strip

separatrices for reduced advection problem

Aref, Stremler & Ponta 2006 JFS
\((\Gamma_1, \Gamma_2, \Gamma_3) = (2, 1, -3)\) in a periodic strip

corresponding vortex motion
\((\Gamma_1, \Gamma_2, \Gamma_3) = (2, 1, -3)\) in a periodic strip
\((\Gamma_1, \Gamma_2, \Gamma_3) = (7, 3, -10)\)

\[ \Xi = L(3 + 2i)/5 \]

separatrices for reduced advection problem

158 stationary vortices

Stremler & Aref

*JFM 392* (1999)
corresponding vortex motion
$N=3$ in a periodic domain also gives relative equilibria

$$2\pi i \ V^* = \Gamma_2 \ \Phi_i(z_1 - z_2) - \Gamma_3 \ \Phi_i(z_3 - z_1)$$

$$2\pi i \ V^* = \Gamma_3 \ \Phi_i(z_2 - z_3) - \Gamma_1 \ \Phi_i(z_1 - z_2)$$

$$2\pi i \ V^* = \Gamma_1 \ \Phi_i(z_3 - z_1) - \Gamma_2 \ \Phi_i(z_2 - z_3)$$

multiply each by $\Gamma_n$ and sum over $n$ \[ S V^* = 0 \]

assume \[ S = \sum_{n=1}^{N} \Gamma_n = 0 \] define \[ w_1 = z_2 - z_3 \]
\[ w_2 = z_3 - z_1 \]
\[ w_3 = z_1 - z_2 \]
\[ = -w_1 - w_2 \]

\[ \Phi_i(w_1 + w_2) - \Phi_i(w_1) = \Phi_i(w_2) \]  

set \( w_2 \) and solve for \( w_1 \)

these configurations are independent of vortex strengths
Summary / Outlook

• period point vortex systems with $N \leq 3$ provide a rich and interesting collection of static configurations and (integrable) dynamics

• can the periodic strip model lend insight into exotic wakes? (Aref, Stremler & Ponta 2006 *JFS*)

• can the periodic parallelogram model lend insight into the development of 2D turbulence?

• looking beyond $N \leq 3$
  - O’Neil: singly periodic vortex statics with large $N$
  - Vlachakis (session IV): the Domm system with $N=4$
an exotic wake vs. time evolution of a point vortex system

Williamson & Roshko (1988) JFS

Aref & Stremler (1996) JFM