Dynamics of Binary Asteroid Pairs

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Apply geometric mechanics and transport calculations to asteroid pairs to calculate, e.g., escape rates.
Motivation

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Dactyl in orbit about Ida, discovered in 1994 during the Galileo mission.
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Time history of the orbit radius (a) and rotation period (b) for a gravitationally interacting sphere and tri-axial ellipsoid of equal mass.
Full Body Problem

■ Full Body Problem (N Bodies)
■ begin with Full Two Body Problem

Applications to...

Planetary Dynamics
Vehicle Control
Chemical Dynamics
Full Body Problem

Relevant to

- asteroid and Kuiper belt binary evolution
- variation of planetary obliquities
- comet nucleus evolution due to outgassing
- close approaches of galaxies
Full Body Problem

galaxy collision
Furthermore

The mathematical description of the FBP and phase space transport phenomena applies to a wide range of physical systems across many scales (chemistry, biology, fluid dynamics, ...)
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□ Phase space transport: For chaotic regimes of motion, the phase space has structures mediating transport (tube and lobe dynamics, ...).
Important Tools

- **Geometric mechanics**: Mechanical systems with symmetry; conserved quantities and reduction.

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- **Phase space transport**: For chaotic regimes of motion, the phase space has structures mediating transport (tube and lobe dynamics, ...).

- Approximate statistical models may be appropriate under certain conditions, e.g., mixing assumptions in chemical and celestial mechanics, etc.
Consider two masses, $m_1$ and $m_2$.

A widely used model: $m_1$ is a sphere.

The normalized and symmetry reduced equations are

$$\ddot{r} + 2\omega \times \dot{r} + \dot{\omega} \times r + \omega \times (\omega \times r) = \frac{\partial U}{\partial r}$$

$$\mathcal{I} \cdot \dot{\omega} + \omega \times \mathcal{I} \cdot \omega = -\mu r \times \frac{\partial U}{\partial r},$$

where

$\omega$ = rotational velocity vector in the body-fixed frame,
$r$ = relative position vector in the body-fixed frame,
$\mathcal{A}$ = attitude tensor of the non-spherical body,
$\mathcal{I}$ = specific inertia tensor of the non-spherical body,
$U$ = gravitational potential of the non-spherical body.
free parameter:  \( \mu = \frac{m_1}{m_1 + m_2} \)
F2BP: Models To Use

- free parameter: \( \mu = \frac{m_1}{m_1 + m_2} \)
- \( \mu \rightarrow 0 \)
  - Particle around asteroid
  - restricted F2BP (RF2BP)
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- \( \mu \rightarrow 0 \)
  
  Particle around asteroid

  restricted F2BP (RF2BP)

- \( \mu \rightarrow 1 \)
  
  Spacecraft around planet
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All the power of geometric mechanics can be brought to bear: symmetry reduction, relative equilibria, energy-momentum method (and its converse), phases (translational and rotational drift and coupling),...
Reduction for the FB2P

- For the F2BP, \( Q = SE(3) \times SE(3) \).
- Material points in a reference configuration \( X_i \).
- Points in the current configuration \( x_i \).
- Given \( ((A_1, r_1), (A_2, r_2)) \in SE(3) \times SE(3) \), related by \( x_1 = A_1X_1 + r_1 \) and \( x_2 = A_2X_2 + r_2 \).
- Lagrangian equals kinetic minus potential energy:

\[
L(A_1, r_1, A_2, r_2) = \frac{1}{2} \int_{B_1} \|\dot{x}_1\|^2 d\mu_1(X_1) + \frac{1}{2} \int_{B_2} \|\dot{x}_2\|^2 d\mu_2(X_2) + \int_{B_1} \int_{B_2} \frac{Gd\mu_1(X_1)d\mu_2(X_2)}{\|x_1 - x_2\|}
\]

\[
= \frac{m_1}{2} \|\dot{r}_1\|^2 + \frac{1}{2} \langle \Omega_1, I_1\Omega_1 \rangle + \frac{m_2}{2} \|\dot{r}_2\|^2 + \frac{1}{2} \langle \Omega_2, I_2\Omega_2 \rangle + \int_{B_1} \int_{B_2} \frac{Gd\mu_1(X_1)d\mu_2(X_2)}{\|A_1X_1 - A_2X_2 + r_1 - r_2\|}.
\]
Reduction for the FB2P

- Reduce by overall translations and rotations.
- $SE(3)$ acts by the diagonal left action on $Q$:
  \[(A, r) \cdot (A_1, r_1, A_2, r_2) = (AA_1, Ar_1 + r, AA_2, Ar_2 + r).\]
- **Momentum map** is the total linear momentum and the total angular momentum.
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  $$(A, r) \cdot (A_1, r_1, A_2, r_2) = (AA_1, Ar_1 + r, AA_2, Ar_2 + r).$$
- **Momentum map** is the total linear momentum and the total angular momentum.
- **Shape space** $Q/G$: one copy of $SE(3)$; coordinatized by the relative attitude $A^T_2A_1 = A^T$ and relative position $A^T_2(r_1 - r_2) = R$.
- General reduction theory says that the reducted equations of motion are in $T(Q/G) \times \mathfrak{g}$ (for velocities) or $(T^*Q)/G \times \mathfrak{g}^*$ (for momenta).

Coupled to equations in $\mathfrak{se}(3)^*$, identified with equations for the body angular and linear momenta of the second rigid body, $\Gamma_2$, $P_2$. 
Reduction for the FB2P

- **Equations of motion** in $T(Q/G)$ (resp. $T^*(Q/G)$) involve $A$, $R$, and their velocities (resp. conjugate momenta $\Gamma$, $P$).

- Coupled to equations in $se(3)^*$, identified with equations for the body angular and linear momenta of the second rigid body, $\Gamma_2$, $P_2$.

- **Shape space**: key to geometric phases that are important for rotational and translational drifts.

- **Reduced Lagrangian**: rewrite $L$ in variables:

  \[
  A = A_2^T A_1, \quad R = A_2^T (r_1 - r_2), \\
  \hat{\Omega} = A_2^T \dot{A}_1, \quad V = A_2^T (\dot{r}_1 - \dot{r}_2),
  \]
which are coordinates on $T(Q/G)$, as well as

$$\hat{\Omega}_2 = A^T_2 \dot{A}_2, \ V_2 = A^T_2 \dot{r}_2,$$

which are coordinates on $\mathfrak{se}(3)$.

**Hamilton’s variational principle** on $T(SE(3) \times SE(3))$ is equivalent to the **reduced variational principle**,

$$\delta \int_a^b l(A, R, \hat{\Omega}, V, \hat{\Omega}_2, V_2) dt = 0,$$

on $\mathbb{R}^{18}$ where the variations are of the form,

$$\delta A = -\hat{\Sigma}_2 A + \hat{\Sigma}, \ \delta R = -\hat{\Sigma}_2 R + S, \ \delta \hat{\Omega} = \hat{\Sigma} - \hat{\Sigma}_2 \hat{\Omega} + \hat{\Omega}_2 \hat{\Sigma},$$

$$\delta V = \dot{S} - \hat{\Sigma}_2 V + \hat{\Omega}_2 S, \ \delta \Omega_2 = \hat{\Sigma}_2 + \Omega_2 \times \Sigma_2, \ \delta V_2 = \dot{S} - \hat{\Sigma}_2 V_2 + \hat{\Omega}_2 S_2.$$
Systematic Structures

- For numerics as well as analysis of stability of relative equilibria (analog of the libration points), the variational and Hamiltonian structures are useful.
- Previous works guessed these structures and missed the variational structure altogether. Using reduction, one derives them in a simple and natural way, one gets the Jacobi integrals naturally, etc.
- Extra symmetries give extra conserved quantities and further reductions (e.g., cylindrical symmetry of one of the bodies).
- Special cases (such as an ellipsoid and a sphere).
Let’s look at an example problem

Restricted (as in restricted 3-body problem) simple case exhibits the basic capture, ejection, collision dynamics.
Restricted Simpler Case

Point mass $P$ moving in the $x$-$y$ plane under the gravitational field of a uniformly rotating elliptical body $m$, without affecting its uniform rotation.

The rotating ($x$-$y$) and inertial ($X$-$Y$) frames.
Equations of motion relative to a rotating Cartesian coordinate frame and appropriately normalized:

\[ \ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \quad \text{and} \quad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}, \]

where

\[ U(x, y) = -\frac{1}{r} - \frac{1}{2}r^2 - \frac{3C_{22}}{r^5} \left( x^2 - y^2 \right) \]

and

\[ r = \sqrt{x^2 + y^2}. \]

Gravity field coefficient \( C_{22} \), the ellipticity, typically varies between 0 and 0.05.

Jacobi integral: \( E = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 \right) - U(x, y). \)
Moving systems approach gives the Lagrangian and Hamiltonian structure and Jacobi integral.

Lagrangian (kinetic minus potential energy) written in the rotating system and with angular velocity normalized to unity, is

\[
L = \frac{1}{2}[(\dot{x} - y)^2 + (x + \dot{y})^2] - V(x, y).
\]

where

\[
V(x, y) = -\frac{1}{r} - \frac{3C_{22}}{r^5}(x^2 - y^2).
\]

Euler–Lagrange equations produce the previous equations and the Legendre transformation gives the Hamiltonian structure, the Jacobi integral, etc.
The Jacobi integral (energy) is an indicator of the type of global dynamics possible.

For energies above a threshold, $E > E_S$, corresponding to symmetric saddle points, movement between the realm near the asteroid (interior realm) and away from the asteroid (exterior realm) is possible. For energies $E \leq E_S$, no such movement is possible.
Multi-scale dynamics: for chaotic regimes of motion, the phase space has structures mediating transport.
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Tube dynamics: On the largest scale, phase space is organized into realms, connected via tubes.
□ **Multi-scale dynamics**: for chaotic regimes of motion, the phase space has structures mediating transport.

□ **tube dynamics**: On the largest scale, phase space is organized into realms, connected via tubes.

□ **lobe dynamics**: In each realm, phase space is organized further into different resonance regions, connected via lobes.
- Slices of energy surface: Poincaré sections $U_i$
- Lobe dynamics: evolution **on** $U_i$
- Tube dynamics: evolution **between** $U_i$
Study Poincaré surface of section on energy surface:

\[ U_i = \Sigma(\mu, E) = \{(x, \dot{x}) | y = 0, \dot{y} = g(x, \dot{x}; \mu, E) > 0\} \]

reducing the system to an area preserving map on the plane,

\[ f_i : U_i \longrightarrow U_i, \]
Transport in Poincaré Section

- Phase space divided into regions $R_i, i = 1, ..., N_R$ bounded by segments of stable and unstable manifolds of unstable fixed points.
Transport between regions is computed via *lobe dynamics*. 

\[ L_{2,1}(1) \quad R_2 \]
\[ q_0 \]
\[ f^{-1}(q_0) \quad f(L_{1,2}(1)) \]
\[ f(L_{2,1}(1)) \]
\[ pi \]
\[ q_1 \]
\[ L_{1,2}(1) \quad R_1 \]
\[ pj \]
We can compute manifolds which naturally divide the phase space into *resonance regions*. Unstable and stable manifolds in **red** and **green**, resp.
Movement Between Resonances

Transport and mixing between regions can be computed.

Four sequences of color coded lobes are shown.
A Poincaré section with $C_{22} = 0.05$ & fixed energy, illustrates the relevance of tube and lobe dynamics.

Choose the section in the exterior region along the positive $x$-axis.

Choose $E = -1.62$, slightly above energy of saddle points along the $x$-axis, such that particles beginning in the exterior region may be ejected from the system or collide with the asteroid.
Exterior Poincaré Sections

(a) Poincaré section for the exterior realm
Exterior Poincaré Sections

- Particles in the exterior region get captured by the asteroid if they lie within the phase space tubes associated with the unstable periodic orbits about either the left or right saddle points. Consider a captured particle to have “collided” with the asteroid if it enters the circle of radius 1 around the origin.
Exterior Poincaré Sections

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- Tube slices on this section: tube slices of collision.
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Tube slices on this section: tube slices of collision.

Particles are ejected if they lie within lobes enclosed by the stable and unstable manifolds of a hyperbolic fixed point at \((+\infty, 0)\)—lobes of ejection.
Exterior Poincaré Sections

□ Transform to Delaunay variables.

(b) Poincaré section in Delaunay variables
The semimajor axis is shown versus the argument of periapse with respect to the rotating asteroid (the body-fixed frame).

Alternate fates of collision and ejection are intimately intermingled.

The number of particles remaining in the fourth quadrant is smaller than that in the other three quadrants, in agreement with observations in Scheere’s work.
Escape and re-capture.
Selected References


For papers, movies, etc., visit the websites:

http://www-personal.engin.umich.edu/~scheeres
http://www.cds.caltech.edu/~shane
http://www.nast-group.caltech.edu/
The End

Typesetting Software: TEX, Textures, \LaTeX, hyperref, texpower, Adobe Acrobat 4.05
Graphics Software: Adobe Illustrator 10.0.1
\LaTeX\ Slide Macro Packages: Wendy McKay, Ross Moore