Resonance Overlap and Transport in the Restricted Three-Body Problem

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Midwest Dynamical Systems Seminar
University of Cincinnati, October 5, 2002
Acknowledgements

☐ C. McCord, K. Meyer
☐ W. Koon, M. Lo, J. Marsden, F. Lekien
☐ G. Gómez, J. Masdemont
☐ C. Jaffé, D. Farrelly, T. Uzer, S. Wiggins
☐ K. Howell, B. Barden, R. Wilson
☐ C. Simó, J. Llibre, R. Martinez
☐ E. Belbruno, B. Marsden, J. Miller
☐ H. Poincaré, J. Moser, C. Conley, R. McGehee
Outline of talk

- **Insight into some dynamical astronomy phenomena can be gained by a restricted three-body analysis**
  - e.g., Jupiter-family comets and scattered Kuiper Belt objects (under Neptune’s control); near-Earth objects
  - By applying dynamical systems methods to the planar, circular restricted three-body problem, several questions regarding these populations may be addressed
- Outline some theoretical ideas
- Several computational results will be shown
- Comparison with observational data is made
- Future directions: other $N$-particle systems
Transport Theory

- Chaotic dynamics $\Rightarrow$ statistical methods

- Transport theory

  - Ensembles of phase space trajectories
    - How long (or likely) to move from one region to another?
    - Determine transition probabilities, correlation functions

  - Applications:
    - Atomic ionization rates
    - Chemical reaction rates
    - Comet and asteroid escape rates, resonance transition probabilities, collision probabilities
Transport Theory

Transport in the solar system

- For objects of interest
  - e.g., Jupiter family comets, near-Earth asteroids, dust

- Identify phase space objects governing transport

- View $N$-body as multiple restricted 3-body problems

- Look at stable and unstable manifold of periodic orbits associated with Lagrange points and mean motion resonances

- Use these to compute statistical quantities
  - e.g., probability of resonance transition, escape rates
We want to answer several questions regarding the transport and origin of some kinds of solar system material:

- How do we characterize the motion of Jupiter-family comets (JFCs) and scattered Kuiper Belt objects (SKBOs)?
- How probable is a Shoemaker-Levy 9-type collision with Jupiter? Or an asteroid collision with Earth (e.g., KT impact 65 Ma)?
- How likely is a transition from outside a planet’s orbit to inside (e.g., the dance of comet Oterma with Jupiter)?

We can answer these questions by considering the phase space.

Harder questions:

- How does impact ejecta get from Mars to Earth?
- How does an SKBO become a comet or an Oort Cloud comet?
- Find features common to all exo-solar planetary systems?
Jupiter Family Comets

- JFCs and lines of constant **Tisserand parameter**,

\[ T = \frac{1}{a} + 2\sqrt{a(1 - e^2)}, \]

an approximation of the Jacobi constant (i.e., \( C = T + \mathcal{O}(\mu) \))
Jupiter Family Comets

**Physical example of intermittency**

- We consider the **historical record** of the comet **Oterma** from 1910 to 1980
  - first in an inertial frame
  - then in a rotating frame
  - a special case of pattern evocation

- Similar pictures exist for many other comets
Jupiter Family Comets

- Rapid transition: outside to inside Jupiter’s orbit.
  - Captured temporarily by Jupiter during transition.
  - Exterior (2:3 resonance) to interior (3:2 resonance).
Oterma’s orbit in rotating frame with some invariant manifolds of the 3-body problem superimposed.
Collisions with Jupiter

- **Shoemaker Levy-9**: similar energy to **Oterma**
  - Temporary capture and collision; came through L1 or L2

Possible Shoemaker-Levy 9 orbit seen in rotating frame (Chodas, 2000)
Scattered Kuiper Belt objects

- Current SKBO locations in black, with some Tisserand values w.r.t. Neptune in red ($T \approx 3$)
Scattered Kuiper Belt Objects

- Seen in inertial space
Observation and numerical experiments show chaotic motion maintaining nearly constant Tisserand parameter in the short-term (i.e., a few Lyapunov times, $\sim 10^2$ to $10^3$ years, cf. Tancredi [1995]).

We approximate the short-timescale motion of JFCs and SKBOs as occurring within an energy shell of the restricted three-body problem.

Several objects may be in nearly the same energy shell, i.e., all have $T$ s.t. $|T - T^*| \leq \delta T$ for some $T^*, \delta T$.

We analyze the structure of an energy shell to determine likely locations of JFCs and SKBOs.
Three-Body Problem

- **Circular restricted 3-body problem**
  - the two primary bodies move in circles; the much smaller third body moves in the gravitational field of the primaries, without affecting them
  - the two primaries could be the Sun and Earth, the Earth and Moon, or Jupiter and Europa, etc.
  - the smaller body could be a spacecraft or asteroid
  - we consider the planar and spatial problems
  - there are five equilibrium points in the rotating frame, places of balance which generate interesting dynamics
Three-Body Problem

- Equations of motion:

\[ \ddot{x} - 2\dot{y} = -U_x^{\text{eff}}, \quad \ddot{y} + 2\dot{x} = -U_y^{\text{eff}} \]

where

\[ U^{\text{eff}} = -\frac{(x^2 + y^2)}{2} - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}. \]

- Have a first integral, the Hamiltonian energy, given by

\[ E(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U^{\text{eff}}(x, y). \]

- Energy manifolds are 3-dimensional surfaces foliating the 4-dimensional phase space.

- This is for the planar problem, but the spatial problem is similar.
Effective potential

In a rotating frame, the equations of motion describe a particle moving in an effective potential plus a Coriolis force (goes back to the work of Jacobi, Hill, etc.)
**Partition the Energy Surface**

- **Restricted 3-body problem (planar)**

- Partition the energy surface: **S, J, X regions**

![Diagram showing the energy surface partitioned into S, J, and X regions.](image-url)
**Stable** and **unstable** manifold tubes

- Control transport through the neck.
For fixed $\mu$, an energy shell (or energy manifold) of energy $\varepsilon$ is

$$\mathcal{M}(\mu, \varepsilon) = \{(x, y, \dot{x}, \dot{y}) \mid E(x, y, \dot{x}, \dot{y}) = \varepsilon\}.$$ 

The $\mathcal{M}(\mu, \varepsilon)$ are 3-dimensional surfaces foliating the 4-dimensional phase space.
Study Poincaré surface of section at fixed energy $\varepsilon$:

$$\Sigma_{(\mu, \varepsilon)} = \{(x, \dot{x}) | y = 0, \dot{y} = f(x, \dot{x}, \mu, \varepsilon) < 0\}$$

reducing the system to an area preserving map on the plane. Motion takes place on the cylinder, $S^1 \times \mathbb{R}$. 

Poincaré surface-of-section and map $P$
Connected chaotic component

- The energy shell has regular components (KAM tori) and irregular components. Large connected irregular component is the "chaotic sea."
Movement among resonances

- The motion within the chaotic sea is understood as the movement of trajectories among resonance regions (see Meiss [1992] and Schroer and Ott [1997]).
Movement among resonances

Schematic of two neighboring resonance regions from Meiss [1992]
Movement among resonances

- Confirmed by numerical computation.
- Shaded region bounded by stable and unstable invariant manifolds of an unstable resonant (periodic) orbit.
Homoclinic tangle

- Unstable/stable manifolds of periodic points understood as the backbone of the dynamics. This is the homoclinic tangle glimpsed by Poincaré.
Resonance region

- Unstable/stable manifolds up to “pip” (cf. Wigging [1992]) denote the boundary of the resonance region.
- Neighboring resonance regions indeed overlap, leading to complicated mixing.
Transport quantities

- **Lobe dynamics**: following intersections of stable and unstable invariant manifolds of periodic orbits (Wiggins et al.)
Transport quantities

- These methods are preferred over the “brute force” solar system calculations seen in the literature since they are based on first principles.

- Reveal generic structures; give deeper insight.
**Obtain rates and probabilities**

- One can compute the rate of escape of asteroids temporarily captured by Mars.
  - Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002]

- Statistical approach
  - similar to chemical dynamics, see Truhlar [1996]

- Consider an asteroid (or other body) in orbit around Mars (perhaps impact ejecta) at a 3-body energy such that it can escape toward the Sun.

- Interested in rate of escape of such bodies at a fixed energy, i.e. $F_{M,S}(t)$
Hill’s Region (PCR3BP)

To fix energy value $E$ is to fix height of plot of $U(x, y)$. Contour plots give 5 cases of Hill’s region.

Case 1: $C > C_1$

Case 2: $C_1 > C > C_2$

Case 3: $C_2 > C > C_3$

Case 4: $C_3 > C > C_4 = C_5$
Escape rates

- Mixing assumption: all asteroids in the Mars region at fixed energy are equally likely to escape.

- Escape rate: \[
\frac{\text{flux out of Mars region}}{\text{Mars region phase space volume}} = \frac{\text{area of escaping orbits}}{\text{area of chaotic region}}
\]
Escape rates

Mars region ($C = 3.000202385$)
Escape rates

- Compare with Monte Carlo simulations of 107,000 particles
  - randomly selected initial conditions at constant energy
Theory and numerical simulations agree well.

- Monte Carlo simulation (dashed) and theory (solid)
Steady state distribution

If the planar, circular restricted three-body problem is **ergodic**, then a statistical mechanics can be built (cf. ZhiGang [1999]).

Recent work suggests there may be regions of the energy shell for which the motion is ergodic, in particular the “chaotic sea” (Jaffé et al. [2002]).

This suggests we compute the **steady state distribution** of some observable for particles in the chaotic sea; a simple method for obtaining the likely locations of any particles within it.
Assuming ergodicity,

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t A(x, y, p_x, p_y) d\tau =
\]

\[
\int A(x, y, p_x, p_y) \frac{C}{|\partial H/\partial p_y|} dp_x dx dy,
\]

where \( A(x, y, p_x, p_y) \) is any physical observable (e.g., semimajor axis), one can finds that the density function, \( \rho(x, p_x) \), on the surface-of-section, \( \Sigma_{(\mu, \varepsilon)} \), is constant.

We can determine the steady state distribution of semi-major axes; define \( N(a) da \) as the number of particles falling into \( a \to a + da \) on the surface-of-section, \( \Sigma_{(\mu, \varepsilon)} \).
SKBOs should be in regions of high density.
Collision Probabilities

- Low velocity impact probabilities
- Assume object enters the planetary region with an energy slightly above L1 or L2
  - e.g., Shoemaker-Levy 9 and Earth-impacting asteroids

![Example Collision Trajectory](image)
Stable and unstable manifold tubes

- Control transport through the neck.
Collision Probabilities

Collision probabilities

- Compute from tube intersection with planet on Poincaré section
- Planetary diameter is a parameter, in addition to $\mu$ and energy $E$
Collision Probabilities

Collision probabilities

Poincare Section: Tube Intersecting a Planet

Py

y

← Diameter of planet →
Collision Probability for Jupiter Family Comets

- **L1**
- **L2**

Energy vs. Collision probability (%)

Collision Probabilities
Collision Probabilities

Collision Probability for Near–Earth Asteroids

Energy (scaled)
Collision probability (%)
Conclusion and Future Work

Transport in the solar system

- Approximate some solar system phenomena using the restricted 3-body problem
- Circular restricted 3-body problem
  - Stable and unstable manifold tubes of libration point orbits can be used to compute statistical quantities of interest
  - Probabilities of transition, collision
- Theory and observation agree

Future studies to involve multiple three-body problems and 3-d.o.f.
References


For papers, movies, etc., visit the website: [http://www.cds.caltech.edu/~shane](http://www.cds.caltech.edu/~shane)

The End