Phase space transport. II

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Apply transport calculations to asteroid pairs to calculate, e.g., capture & escape rates.
Motivation

- Apply transport calculations to asteroid pairs to calculate, e.g., capture & escape rates.

Dactyl in orbit about Ida, discovered in 1994 during the *Galileo* mission.
Motivation

- Movie of simple model of Ida & Dactyl
- Small hard sphere around rotating elliptical asteroid
- Nonmerging collisions modeled as bounces
  (work with E. Kanso)
In This Talk...

- **Part I. Restricted F2BP phase space**
  - Dependence on energy
  - Tube + lobe dynamics
  - Modeling collisions

- **Part II. Transport using set oriented methods**
  - Transport problem described
  - Two computational techniques
    (a) Invariant manifolds
    (b) Almost-invariant sets
  - Extensions and future work
Part I

**Restricted Full Two Body Problem**

- Consider two masses: $m_1$ (sphere) & $m_2$ (ellipse)

\[
\frac{m_1}{m_2} \to 0
\]

Particle around asteroid

restricted F2BP (**RF2BP**)

- Restricted (as in restricted 3-body problem) simple case exhibits the basic capture, ejection, collision dynamics (see Koon, Marsden, Ross, Lo, Scheeres [2003])

- Can include bouncing, sticking, ... (see Kanso [2003])
Point mass $P$ moving in the $x$-$y$ plane under the gravitational field of a uniformly rotating elliptical body $m$, without affecting its uniform rotation.

The rotating ($x$-$y$) and inertial ($X$-$Y$) frames.
Equations of motion relative to a rotating Cartesian coordinate frame and appropriately normalized:

\[ \ddot{x} - 2\dot{y} = -\frac{\partial U}{\partial x} \quad \text{and} \quad \ddot{y} + 2\dot{x} = -\frac{\partial U}{\partial y}, \]

where

\[ U(x, y) = -\frac{1}{r} - \frac{1}{2}r^2 - \frac{3C_{22}}{r^5}(x^2 - y^2), \]

and

\[ r = \sqrt{x^2 + y^2}. \]

Energy integral (Jacobi integral):

\[ E = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + U(x, y). \]
Gravity field coefficient $C_{22}$, the **ellipticity**, varies between 0 and 0.05.

e.g., Ida: $\beta \approx 0.43, C_{22} \approx 0.04$
Energy indicates type of global dynamics.

Is movement between the exterior and asteroid realms possible?

\[ E < E_S \]

\[ E > E_S \]
Phase Space Structure

- Multi-scale dynamics
  - Coarse level: tube dynamics between realms
  - Fine level: lobe dynamics within realms
Phase Space Structure

- Slices of energy surface: Poincaré sections $U_i$
- Tube dynamics: evolution between $U_i$
- Lobe dynamics: evolution on $U_i$
Suppose $E < E_S$; energy surface $\mathcal{M}_E$

Asteroid and exterior realms not connected

Poincaré map in exterior realm: area and orientation preserving map on $M \subset \mathbb{R}^2$,

$$f : M \longrightarrow M$$

where

$$M = \mathcal{M}_E \cap \{x = 0, \dot{x} > 0\}$$

with coordinates $(y, p_y)$, equiv., $(r, p_r)$, on $M$
Particles are **ejected** if they lie within lobes enclosed by the stable and unstable manifolds of a hyperbolic fixed point at $(+\infty, 0)$—**lobes of ejection**.
Lobe Dynamics

Numerical simulation using MANGEN
Curves can be followed to very high accuracy
Simulations use MANGEN (Coulliette & Lekien)

Adaptive conditioning of curves based on curvature.
Tube + Lobe Dynamics

- Suppose $E > E_S$
- Exterior and asteroid realms connected via tubes
- In exterior realm, some tubes lead to collision (others lead away from collision)
- Tube + lobe dynamics = Alternate fates of collision and ejection are intimately intermingled.
Tube + Lobe Dynamics

- Tubes leading to collision with asteroid

Position space projection

Motion on $M$
Tube + Lobe Dynamics

- Tubes leading to collision with asteroid
- **plus** tubes coming from collision, e.g., liberated particles

### Diagram

Position space projection

Motion on $M$
Tube + Lobe Dynamics

- Escape and re-capture.
If bouncing is modeled, dynamics is more complicated.

Upon bouncing, particle moves to new energy surface.

Work in progress with E. Kanso.
Transport using set oriented methods

Describe transport of phase points on a $k$-dimensional manifold $M$
Part II

- *Transport using set oriented methods*

  - Describe transport of phase points on a $k$-dimensional manifold $M$
  - $M$ could be, e.g., the ocean surface, an energy shell, or a Poincaré surface-of-section
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Transport using set oriented methods

- Describe transport of phase points on a $k$-dimensional manifold $M$

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- We look first at $k = 2$ for autonomous systems

Consider a volume- and orientation-preserving map

\[ f : M \rightarrow M, \]

on some compact set \( M \subset \mathbb{R}^2 \) with volume measure \( \mu \). e.g., \( f \) may be a discretization of an autonomous flow.
Consider a volume- and orientation-preserving map

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**Example:** \( f \) may be a discretization of an autonomous flow.

\( M \) is **partitioned into regions** \( R_i, i = 1, \ldots, N_R \), such that

\[
M = \bigcup_{i=1}^{N_R} R_i \quad \text{and} \quad \mu(R_i \cap R_j) = 0 \quad \text{for} \ i \neq j.
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Initially, \( R_i \) is uniformly covered with species \( S_i. \)
i.e., Species type indicates where a point was initially.
Describe the distribution of species $S_i$ throughout the regions $R_j$ at any future iterate $n > 0$. 
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Transport Quantities

Quantities of interest:

\[ T_{i,j}(n) \equiv \text{the total amount of species } S_i \text{ contained in region } R_j \text{ immediately after the } n\text{-th iterate} \]
\[ = \mu(f^{-n}(R_j) \cap R_i) \]
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Our goal:

Compute the \( T_{i,j}(n) \) up to some \( n_{\text{max}} \)
Compute approaches

- Compare & combine two computational approaches
Computational Approaches

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1) **Invariant manifolds** of fixed points, lobe dynamics; co-dimension one objects which bound regions, etc.

MANGEN: Manifold Generation, Lekien etc
Computational Approaches

- Compare & combine two computational approaches

1) **Invariant manifolds** of fixed points, lobe dynamics; co-dimension one objects which bound regions, etc. MANGEN: Manifold Generation, Lekien etc

2) **Set oriented methods**, almost-invariant sets; direct computation of regions GAIO: Global Analysis of Invariant Objects, Dellnitz etc
Example problem: test particles in gravity field of two masses, $m_1$ and $m_2$, in circular orbit, i.e., the planar, circular restricted three-body problem with $\frac{m_2}{m_1} \approx 10^{-3}$.

Reduce to 2D map via Poincaré surface-of-section.
Poincaré map $f : M \rightarrow M$ has regular and irregular components. Large connected irregular component, the “chaotic sea.”
To understand the transport of points under the Poincaré map $f$, we consider the invariant manifolds of unstable fixed points.

Let $p_i, i = 1, \ldots, N_p$, denote a collection of saddle-type hyperbolic fixed points for $f$. 
Transport on $\mathcal{M}$

- Local pieces of unstable and stable manifolds

Unstable and stable manifolds in **red** and **green**, resp.
Intersection of unstable and stable manifolds define boundaries.
These boundaries divide phase space into regions, $R_i, i = 1, \ldots, N_R$. 

The diagram shows a phase space with regions $R_1, R_2, R_3, R_4, R_5$, labeled points $q_1, q_2, q_3, q_4, q_5, q_6$, and arrows indicating the flow or transport within the phase space.
Local transport: across a boundary
consider small sets bounded by stable & unstable mfds
They map from entirely in one region to another under one iteration of $f$

$L_{1,2}(1)$ and $L_{2,1}(1)$ are called turnstile lobes
MANGEN: evolution of a lobe of species $S_1$ into $R_2$
Global transport between regions \((T_{i,j}(n))\) is completely described by the dynamical evolution of lobes.
**Overview**

- Partition phase space into **loosely coupled regions**
  
  \[ R_i, i = 1, \ldots, N_R, \]

- Probability is small for a point in a region to leave in a short time under \( f \).

- These **almost-invariant sets** (AIS’s) define macroscopic structures preserved by the dynamics.

- The transport, \( T_{i,j}(n) \), between almost-invariant sets can then be determined.
Almost-Invariant Sets

1) discretize the phase space into boxes; model boxes as the vertices and transitions between boxes as edges of a directed graph
2) use graph partitioning methods to divide the vertices of the graph into an optimal number of parts such that each part is highly coupled within itself and only loosely coupled with other parts.
Almost-Invariant Sets

3) by doing so, we can obtain AIS’s and analyze transport between them
Almost-Invariant Sets

Box Formulation

- Create a fine box partition of the phase space $\mathcal{B} = \{B_1, \ldots, B_q\}$, where $q$ could be $10^7+$.
- Consider a (weighted) $q$-by-$q$ transition matrix, $P$, for our dynamical system, where

$$P_{ij} = \frac{\mu(B_i \cap f^{-1}(B_j))}{\mu(B_i)},$$

the transition probability from $B_i$ to $B_j$.
- $P$ is an approximation of our dynamical system via a finite state Markov chain.
Almost-Invariant Sets

Graph Formulation and Partitioning

$P$ has a corresponding graph representation where nodes of the graph correspond to boxes $B_i$. 

Using the Underlying Graph (Froyland-D. 2003, D.-Preis 2002) Boxes are vertices. Coarse dynamics represented by edges. Use graph theoretic algorithms in combination with the multilevel structure.
Almost-Invariant Sets

- If $P_{ij} > 0$, then there is an edge between nodes $i$ and $j$ in the graph with weight $P_{ij}$.
- Partitioning into AIS’s becomes a problem of finding a minimal cut of this graph.
Almost-Invariant Sets

- AIS’s correspond with key dynamical features
- More refined methods like MANGEN can pick up details

The phase space is divided into several invariant and almost-invariant sets.
Using the box formulation and GAIO, the $T_{i,j}(n)$ can be computed for large $n$. Agrees with MANGEN result.
To speed the computation, box refinements are performed where transport related structures, e.g., lobes, are located.
The merging of statistical and geometric approaches yields a very powerful tool.
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Example problem: restricted 3-body problem.

Both find the same regions

AIS’s, statistical features, are identified with regions, geometric features
Theoretically, transport between regions determined by images and pre-images of lobes.
Summary & Conclusion

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- MANGEN computes only region & lobe boundaries; Stretching of boundaries, memory limits;
  \[ n_{\text{max}} \approx 20 \]
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- **MANGEN** computes only region & lobe boundaries; Stretching of boundaries, memory limits;
  \[ n_{\text{max}} \approx 20 \]

- **GAIO** uses transition matrix between many boxes; Resolution limited by max box number; aided by box refinement along lobes
  \[ n_{\text{max}} \approx 50 \]
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\[ n_{\text{max}} \approx 50 \]

Same values for \( T_{i,j}(n) \) over common time window.
Future Directions

- AIS & lobe dynamics in 3D+, e.g., astronomy, chemistry
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- AIS for time dependent systems? e.g., ocean dynamics
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- **Numerical algorithms are crucial:**
  - GAIO for coarse picture, transport calculations
  - MANGEN to refine on regions of interest
  ⇒ important for precision navigation
Future Directions

- AIS & lobe dynamics in 3D+, e.g., astronomy, chemistry
- AIS for time dependent systems? e.g., ocean dynamics
- **Numerical algorithms are crucial:**
  - GAIO for coarse picture, transport calculations
  - MANGEN to refine on regions of interest
  ⇒ important for precision navigation
- **Merge techniques into single package:**
  - Box formulation, graph algorithms
  - Co-dimension one objects
  - Adaptive conditioning based on curvature


• Ross, S.D. [2003], Statistical theory of interior-exterior transition and collision probabilities for minor bodies in the solar system. *International Conference on Libration Point Orbits and Applications, Girona, Spain*.


For papers, movies, etc., visit the websites:
http://www.cds.caltech.edu/~shane
http://www.nast-group.caltech.edu/
The End

Typesetting Software: TEX, Textures, LATEX, hyperref, texpower, Adobe Acrobat 4.05
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