Lagrangian coherent structures in fluid dynamics

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MultiSTEPS: MultiScale Transport in Environmental & Physiological Systems, www.multisteps.esm.vt.edu
Lagrangian coherent structures – coherent structures moving with the fluid

Some vortex examples from geophysical fluids

- hurricanes
- tornados
- eddies

Why Lagrangian coherent structures?
Natural flow visualization
Insight for design and control
Application to fluid and non-fluid systems
Lagrangian coherent structures

Two strategies from *dynamical systems*:
(1) Identify their `skeleton’ or boundaries
(2) Identify the structures *directly*
Lagrangian coherent structures

Two strategies from *dynamical systems*:
(1) Identify their `skeleton’ or *boundaries*
(2) Identify the structures *directly*

For (1):
For time-periodic or steady fluid velocity fields, identify periodic orbits + their stable & unstable manifolds.

But for *realistic* flows,
Time-independent and aperiodic, data-driven, finite-time, so need something else?

For (2):
Discretize the *flow map* over timescale of interest;
determine coherent regions via eigenmodes, graph methods
Atmosphere: Antarctic polar vortex
Atmosphere: Antarctic polar vortex
Atmosphere: continental U.S.
Periodic velocity field

- If \( \mathcal{M} \) = fluid domain, the flow map,

\[
\phi_{t+T} : \mathcal{M} \rightarrow \mathcal{M},
\]

takes points \( x \mapsto \phi_{t+T}(x) \) to their location after time \( T \).
Periodic velocity field

- Suppose our velocity field is periodic with period $T$ and we consider the flow map $f = \phi^{t+T}$ over one period.

- To understand the transport of points under the map $f$, consider **invariant manifolds of saddle fixed points**.

- Let $p_i, i = 1, \ldots, N_p$, denote a collection of saddle-type hyperbolic fixed points for $f$. 
Partition phase space into coherent regions

- Natural way to partition phase space
  - Pieces of $W^u(p_i)$ and $W^s(p_i)$ partition $\mathcal{M}$.

Unstable and stable manifolds in **red** and **green**, resp.
Partition phase space into coherent regions

- Intersection of unstable and stable manifolds define boundaries.
Partition phase space into coherent regions

- These boundaries divide the phase space into **coherent regions**.
Label mobile subregions: ‘atoms’ of transport

- Can label mobile subregions based on their past and future whereabouts under one iterate of the map, e.g., (..., $R_4, R_4, R_1, [R_1], R_2, \ldots$)
Lobe dynamics: transport across a boundary

- \( U[f^{-1}(q), q] \cup S[f^{-1}(q), q] \) forms boundary of two lobes; one in \( R_1 \), labeled \( L_{1,2}(1) \), or equivalently \( ([R_1], R_2) \), where \( f(([R_1], R_2)) = (R_1, [R_2]) \), etc. for \( L_{2,1}(1) \)
Lobe dynamics: transport across a boundary

- Under one iteration of $f$, only points in $L_{1,2}(1)$ can move from $R_1$ into $R_2$ by crossing $B$, etc.
- The two lobes $L_{1,2}(1)$ and $L_{2,1}(1)$ are called a **turnstile**.
Lobe dynamics: transport across a boundary

Essence of lobe dynamics: dynamics associated with crossing a boundary is reduced to the dynamics of turnstile lobes associated with the boundary.
Extending to realistic flows

- Data-driven, finite-time, aperiodic setting
- How do we get at transport?
- Recall the flow, \( x \mapsto \phi_{t_0+T}(x) \)
Identify regions of high sensitivity of initial conditions

- Small initial perturbations $\delta x(t)$ grow like
  \[
  \delta x(t + T) = \phi^{t+T}_{t}(x + \delta x(t)) - \phi^{t+T}_{t}(x) = \frac{d\phi^{t+T}_{t}(x)}{dx} \delta x(t) + O(\|\delta x(t)\|^2)
  \]
Identify regions of high sensitivity of initial conditions

- Small initial perturbations $\delta x(t)$ grow like

\[
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\]
Invariant manifold analogs: FTLE-LCS approach

- The finite-time Lyapunov exponent (FTLE),
  \[ \sigma^T_t(x) = \frac{1}{|T|} \log \left| \frac{d\phi^{t+T}_t(x)}{dx} \right| \]
  measures the maximum stretching rate over the interval \( T \) of trajectories starting near the point \( x \) at time \( t \)

- Ridges of \( \sigma^T_t \) are candidate hyperbolic codim-1 surfaces; finite-time analogs of stable/unstable manifolds; Lagrangian coherent structures\(^1\)

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\(^1\) cf. Bowman, 1999; Haller & Yuan, 2000; Haller, 2001; Shadden, Lekien, Marsden, 2005
Invariant manifold analogs: FTLE-LCS approach
Invariant manifold analogs: FTLE-LCS approach
Invariant manifold analogs: FTLE-LCS approach

- We can define the FTLE for Riemannian manifolds\(^2\)

\[
\sigma_t^T(x) = \frac{1}{|T|} \ln \left\| D\phi_t^{t+T} \right\| = \frac{1}{|T|} \ln \left( \max_{y \neq 0} \frac{\left\| D\phi_t^{t+T}(y) \right\|}{\|y\|} \right)
\]

with \(y\) a small perturbation in the tangent space at \(x\).

\(^2\)Lekien & Ross [2010] Chaos
Hurricanes and lobe dynamics

Andrea, first storm of 2007 hurricane season

Hurricanes and lobe dynamics

Andrea at one snapshot; LCS shown (orange = repelling, blue = attracting)
Hurricanes and lobe dynamics

orange = repelling (stable manifold), blue = attracting (unstable manifold)
Hurricanes and lobe dynamics

orange = repelling (stable manifold), blue = attracting (unstable manifold)
Hurricanes and lobe dynamics

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out
Hurricanes and lobe dynamics

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out
Hurricanes and lobe dynamics

Sets behave as lobe dynamics dictates
Invasive species riding coherent structures

Hurricane Ivan (2004) brought new crop disease (soybean rust) to U.S.

From Rio Cauca region of Colombia

Disease extent

Cost of invasive organisms is $137 billion per year in U.S.
Atmospheric transport of microorganisms

e.g., *Fusarium*

- Spore production, release, escape from surface
- Long-range transport (time-scale hours to days)
- Deposition, infection efficiency, host susceptibility

Isard & Gage [2001]
Atmospheric transport network relevant for aeroecology

Skeleton of large-scale horizontal transport

relevant for large-scale spatiotemporal patterns of important biota e.g., plant pathogens

orange = repelling LCSs, blue = attracting LCSs
Aerial sampling:
40 m – 400 m altitude
Atmospheric transport network relevant for aeroecology
UAVs and ground-level sampler → Colonies of *Fusarium* → Single-spored cultures

→ Living culture collection

PCR, sequencing, and BLAST searches against FUSARIUM-ID and GenBank

Morphology-based verification
Concentration of *Fusarium* spores (number/m$^3$) for samples from 100 flights conducted between August 2006 and March 2010.
Concentration of *Fusarium* spores (number/m$^3$) for samples from 100 flights conducted between August 2006 and March 2010.
Punctuated changes: correlated to LCS passage?

Detected concentration of Fusarium at sampling location

(time)
Detected concentration of Fusarium at sampling location

Punctuated changes: correlated to LCS passage?
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Detected concentration of Fusarium at sampling location

Sampling location
Punctuated changes: correlated to LCS passage?

Detected concentration of Fusarium at sampling location over time.
Time series of concentration \( \{(t_0, C_0), \ldots, (t_{N-1}, C_{N-1})\} \)
LCS passage times: orange = repelling LCSs, blue = attracting
Spore concentration (spores/m³)

00:00 12:00 00:00 12:00 00:00 12:00

Compare: are there patterns?
Summary of Hypothesis Testing

- Of 100 samples, only 73 sample pairs within 24 hours

- Of those, 16 show punctuated changes in the concentration of *Fusarium*

- Punctuated change $\Rightarrow$ repelling LCS passage *70% of the time* ($p = 0.0017$)

- Punctuated changes were significantly associated with the movement of a repelling LCS

- Correlation poor for attracting LCS: punctuated change $\Rightarrow$ attracting LCS passage *37% of the time* ($p = 0.33$)
Example: Filament bounded by repelling LCS

12:00 UTC 1 May 2007
15:00 UTC 1 May 2007
18:00 UTC 1 May 2007
Example: Filament bounded by repelling LCS

(a) (b) (c)

100 km

Time

Spoconcentration (spores/m³)

00:00 12:00 00:00 12:00 00:00 12:00


12:00 UTC 1 May 2007 15:00 UTC 1 May 2007 18:00 UTC 1 May 2007
Direct computation of coherent sets

- Take probabilistic point of view
- Partition phase space into loosely coupled regions

Coherent sets \(\approx\) “Leaky” regions with a long residence time

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phase space is divided into several invariant and almost-invariant sets.

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\(^3\)work of Dellnitz, Junge, Deuflhard, Froyland, Schütte, et al
Direct computation of coherent sets

- Create box partition of phase space $\mathcal{B} = \{B_1, \ldots B_q\}$, with $q$ large
- Consider a $q$-by-$q$ transition (Ulam) matrix, $P$, for our dynamical system, where

\[
P_{ij} = \frac{m(B_i \cap \phi^{-1}(B_j))}{m(B_i)},
\]

the transition probability from $B_i$ to $B_j$ using $\phi = \phi_{t+T}^t$

- $P$ approximates the flow map $\phi_{t+T}^t$ via a finite state Markov chain.
Direct computation of coherent sets

- A set $B$ is called \textit{almost-invariant} over the interval $[t, t + T]$ if

  $$\rho(B) = \frac{m(B \cap \phi^{-1}(B))}{m(B)} \approx 1.$$ 

- Can maximize value of $\rho$ over all possible combinations of sets $B \in \mathcal{B}$.

- In practice, ALIs or relatedly, almost-cyclic sets (ACSs), identified via \textbf{eigenvectors} (of eigenvalues with $|\lambda| \approx 1$) of $P$ or graph-partitioning...
Identifying coherent sets

- Consider lid-driven cavity flow system with system parameter $\tau_f$
- For $\tau_f > 1$, periodic points and manifolds exist
- Coherent set boundaries are manifolds of periodic points
- Known previously\(^4\) and applies to more general objects than periodic points, i.e. normally hyperbolic invariant manifolds (NHIMs)

Identifying coherent sets

Poincaré section for $\tau_f < 1 \Rightarrow$ no obvious structure!

- For $\tau_f < 1$, no periodic orbits of low period known
- Is the phase space featureless?

- Consider transition matrix $P_t^{t+\tau_f}$ induced by Poincaré map $\phi_t^{t+\tau_f}$
Identifying coherent sets

Top eigenvectors for $\tau_f = 0.99$ reveal hierarchy of phase space structures
Identifying coherent sets

- Three almost-cyclic coherent sets (ACSs) of period 3
- ACSs effectively replace compact region bounded by saddle manifolds
- Also: we see a dynamical remnant of the global ‘stable and unstable manifolds’ of the saddle points, even there are no more saddle points
Identifying coherent sets

Almost-cyclic sets stirring the surrounding fluid like ‘ghost rods’ — works even when periodic orbits are absent!

Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$
Identifying coherent sets

Braid of ACSs gives lower bound of entropy via Thurston-Nielsen
— One only needs approximately cyclic blobs of fluid
— Even though the theorems require exactly periodic points!
Movie shows change in eigenvector along branch marked with ‘⁻→⁻’ above (from a to f), as $\tau_f$ decreases $\Rightarrow$
Coherent sets in the atmosphere

- Coherent sets during 24 hours starting 09:00 1 May 2007
Final words on coherent sets & transport

- What are the robust descriptions of coherent sets and transport which work in aperiodic, finite-time settings?

- Methods for finding boundaries of coherent sets and coherent sets themselves are fruitful for illuminating structure and transport.

- Lobe dynamics, finite-time symbolic dynamics...

- In analogy with point vortices, can we find equations of motion for generalized coherent sets and their influence on each other?
Main Papers:

- Tallapragada, Ross, Schmale [2011] Lagrangian coherent structures are associated with fluctuations in airborne microbial populations. Chaos 21, 033122.