Geometric and probabilistic descriptions of chaotic phase space transport

Shane Ross
Dept. of Engineering Science and Mechanics, Virginia Tech
www.shaneross.com


North Carolina State University, Differential Equations Seminar
Department of Mathematics, November 9, 2011

MultiSTEPS: MultiScale Transport in Environmental & Physiological Systems, www.multisteps.esm.vt.edu
Motivation: application to real data

- Many systems defined from data or large-scale simulations — experimental measurements, observations
- e.g., from fluid dynamics, biology, social sciences

- Data-based, aperiodic, finite-time, finite resolution — in general, no fixed points, periodic orbits, or other invariant sets (or their stable and unstable manifolds) to organize phase space
Motivation: application to real data

• Perhaps can find appropriate analogs to the objects; adapt previous results to this setting

• Try some numerical explorations; see what merit furthers study
Chaotic phase space transport via lobe dynamics

Suppose our dynamical system is a discrete map

\[ f : \mathcal{M} \rightarrow \mathcal{M}, \]

e.g., \( f = \phi_{t+T}^t \), flow map of time-periodic vector field and \( \mathcal{M} \) is a differentiable, orientable, two-dimensional manifold e.g., \( \mathbb{R}^2, S^2 \)

To understand the transport of points under the \( f \), consider invariant manifolds of unstable fixed points

- Let \( p_i, i = 1, \ldots, N_p \), denote saddle-type hyperbolic fixed points of \( f \).

\(^1\)Following Rom-Kedar and Wiggins [1990]
Partition phase space into regions

- Natural way to partition phase space
  - Pieces of $W^u(p_i)$ and $W^s(p_i)$ partition $\mathcal{M}$.

Unstable and stable manifolds in red and green, resp.
Partition phase space into regions

- Intersection of unstable and stable manifolds define **boundaries**.
Partition phase space into regions

- These boundaries divide the phase space into regions.
Label mobile subregions: ‘atoms’ of transport

- Can label mobile subregions based on their past and future whereabouts under one iterate of the map, e.g., \( \ldots, R_4, R_4, R_1, [R_1], R_2, \ldots \)
Lobe dynamics: transport across a boundary

\[ W^u[f^{-1}(q), q] \cup W^s[f^{-1}(q), q] \] forms boundary of two lobes; one in \( R_1 \), labeled \( L_{1,2}(1) \), or equivalently \( ([R_1], R_2) \), where \( f(([R_1], R_2)) = (R_1, [R_2]) \), etc. for \( L_{2,1}(1) \)
Lobe dynamics: transport across a boundary

- Under one iteration of $f$, only points in $L_{1,2}(1)$ can move from $R_1$ into $R_2$ by crossing their boundary, etc.
- The two lobes $L_{1,2}(1)$ and $L_{2,1}(1)$ are called a **turnstile**.
Lobe dynamics: transport across a boundary

Essence of lobe dynamics: dynamics associated with crossing a boundary is reduced to the dynamics of turnstile lobes associated with the boundary.
Identifying atoms of transport by itinerary

In a complicated system, can still identify manifolds ...

Unstable and stable manifolds in red and green, resp.
Identifying atoms of transport by itinerary

... and lobes

Significant amount of fine, filamentary structure.
Identifying atoms of transport by itinerary

e.g., with three regions \( \{ R_1, R_2, R_3 \} \), label lobe intersections accordingly.

- Denote the intersection \(( R_3, [R_2] ) \cap ([R_2], R_1)\) by \(( R_3, [R_2], R_1)\)
Identifying atoms of transport by itinerary

$(R_3,[R_2],R_1)$

Longer itineraries...
Identifying atoms of transport by itinerary

... correspond to smaller pieces of phase space; horseshoe dynamics, etc
Lobe dynamics intimately related to transport

$n = 0$

$n = 1$

$n = 2$

$n = 3$

$n = 5$

$n = 7$
Lobe Dynamics: example

- Restricted 3-body problem: chaotic sea has unstable fixed points.
Compute a boundary

- $R_1$ and $R_2$ are segments of $R$ defining the axis of symmetry as the natural choice for the pips between $L$-turnstile lobes to be self-intersecting turnstile discussed in Sec. 3.1. A schematic of this situation is shown in Fig. 4. In this case we define the turnstile lobes $B = U[p, q] \cup S[p, q]$.

- More points are added where curvatures of the manifold are large, and lobes. The first intersection of $L$-turnstile lobes to be self-intersecting turnstile discussed in Sec. 3.1. A schematic of this situation is shown in Fig. 4. In this case we define the turnstile lobes $B = U[p, q] \cup S[p, q]$.

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Transport btwn Two Regions

- The evolution of a lobe of species $S_1$ into $R_2$

Transport between Two Regions

Species Distribution: Species $S_1$ in Region $R_2$

- $F_{1,2} =$ flux of species $S_1$ into region $R_2$ on the $n$th iterate
- $T_{1,2} =$ total amount of $S_1$ contained in $R_2$ immediately after the $n$th iterate
Lobe dynamics: fluid example

Fluid example: time-periodic Stokes flow

Lid-driven cavity flow

- Model for microfluidic mixer
- System has parameter $\tau_f$, which we treat as a bifurcation parameter
  - critical point $\tau_f^* = 1$; above and next few slides show $\tau_f > 1$

Computations by Mohsen Gheisarieha and Mark Stremler (Virginia Tech)
Lobe dynamics: fluid example

Structure associated with saddles of Poincaré map

some invariant manifolds of saddles
Lobe dynamics: fluid example

Can consider transport via lobe dynamics

pips, regions and lobes labeled
Stable/unstable manifolds and lobes in fluids

material blob at $t = 0$
Stable/unstable manifolds and lobes in fluids

material blob at $t = 5$
Stable/unstable manifolds and lobes in fluids

some invariant manifolds of saddles
Stable/unstable manifolds and lobes in fluids

material blob at $t = 10$
Stable/unstable manifolds and lobes in fluids

material blob at $t = 15$
Stable/unstable manifolds and lobes in fluids

material blob and manifolds
Stable/unstable manifolds and lobes in fluids

material blob at $t = 20$
Stable/unstable manifolds and lobes in fluids

material blob at $t = 25$
Stable/unstable manifolds and lobes in fluids

- Saddle manifolds and lobe dynamics provide template for motion
Stable/unstable manifolds and lobes in fluids

Concentration variance; a measure of homogenization

- Homogenization has two exponential rates: slower one related to lobes
- Fast rate due to braiding of ‘ghost rods’
Stirring fluids with solid rods

- **turbulent mixing**
  - spoon in coffee

- **laminar mixing**
  - 3 ‘braiding’ rods in glycerin
Topological chaos through braiding of stirrers

- Topological chaos is ‘built in’ the flow due to the topology of boundary motions

\[ R_N : \text{2D fluid region with } N \text{ stirring ‘rods’} \]
- stirrers move on periodic orbits
- stirrers = solid objects or fluid particles
- stirrer motions generate diffeomorphism \( f : R_N \to R_N \)
- stirrer trajectories generate braids in 2+1 dimensional space-time
Thurston-Nielsen classification theorem

- A stirrer motion $f$ is isotopic to a stirrer motion $g$ of one of three types
  (i) finite order (f.o.): the $n$th iterate of $g$ is the identity
  (ii) pseudo-Anosov (pA): $g$ has dense orbits,
  (iii) reducible: $g$ contains both
  f.o. and pA regions
- $h_{TN}$ computed from ‘braid word’, e.g., $\sigma_{-1}\sigma_2$
- $\log(\lambda_{PF}(A))$ provides a lower bound on the
  true topological entropy
- i.e., non-trivial material lines grow like $\ell \sim \ell_0 \lambda^n$, where $\lambda \geq \lambda_{TN}$
Topological chaos in a viscous fluid experiment

finite order

pseudoAnosov

\( R_+ \)

\( L_+ \)

\( R_+ \)

\( L_- \)

\( \sigma_1 \)

\( \sigma_2 \)

\( \sigma_1^{-1} \)

\( \sigma_2 \)
Identifying ‘ghost rods’: periodic points

- For $\tau_f > 1$, groups of elliptic and saddle periodic points of period 3 — streamlines around groups resemble fluid motion around a solid rod $\Rightarrow$
- At $\tau_f = 1$, points merge into parabolic points
- Below $\tau_f < 1$, periodic points vanish
Identifying ‘ghost rods’: periodic points

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  — streamlines around groups resemble fluid motion around a solid rod $\Rightarrow$
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Identifying ‘ghost rods’: periodic points

- Periodic points of period 3 ⇒ act as ‘ghost rods’
- Their braid has $h_{TN} = 0.96242$ from TNCT
- Actual $h_{flow} \approx 0.964$
- ⇒ $h_{TN}$ is an excellent lower bound
Consider $\tau_f < 1$
Identifying ‘ghost rods’?

Poincaré section for $\tau_f < 1 \Rightarrow$ no obvious structure!

- Note the absence of any elliptical islands
- No periodic orbits of low period were found
- Is the phase space featureless?
Almost-invariant set (AIS) approach

- Take probabilistic point of view
- Partition phase space into \textit{loosely coupled regions}

Almost-invariant sets $\approx$ ‘leaky’ regions with a long residence time\textsuperscript{2}

\textsuperscript{2}Dellnitz, Junge, Koon, Lekien, Lo, Marsden, Padberg, Preis, Ross, Thiere [2005] Int. J. Bif. Chaos
Almost-invariant set (AIS) approach

- Create box partition of phase space $\mathcal{B} = \{B_1, \ldots B_q\}$, with $q$ large
- Consider a $q$-by-$q$ transition (Ulam) matrix, $P$, for our dynamical system, where

$$P_{ij} = \frac{m(B_i \cap f^{-1}(B_j))}{m(B_i)},$$

the transition probability from $B_i$ to $B_j$ using, e.g., $f = \phi^t_t + T$

- $P$ approximates our dynamical system via a finite state Markov chain.
Almost-invariant set (AIS) approach

- A set $B$ is called almost invariant over the interval $[t, t + T]$ if
  $$\rho(B) = \frac{m(B \cap \phi^{-1}(B))}{m(B)} \approx 1.$$  
- Can maximize value of $\rho$ over all possible combinations of sets $B \in \mathcal{B}$.
- In practice, AIS or relatedly, almost-cyclic sets (ACS), identified via eigenvectors (of eigenvalues with $|\lambda| \approx 1$) of $P$ or graph-partitioning.
- Appropriate for non-autonomous, aperiodic, finite-time settings.
Identifying ‘ghost rods’: almost-cyclic sets

- Return to $\tau_f > 1$ case, where periodic points and manifolds exist
- Agreement between AIS boundaries and manifolds of periodic points
- Known previously\(^3\) and applies to more general objects than periodic points, i.e. normally hyperbolic invariant manifolds (NHIMs)

Identifying ‘ghost rods’: almost-cyclic sets

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Identifying ‘ghost rods’: almost-cyclic sets

Poincaré section for $\tau_f < 1 \Rightarrow$ no obvious structure!

- Return to $\tau_f < 1$ case, where no periodic orbits of low period known
- Is the phase space featureless?

- Consider transition matrix $P_{t+\tau f}^t$ induced by Poincaré map $\phi_t^{t+\tau f}$
Identifying ‘ghost rods’: almost-cyclic sets

Top eigenvectors for $\tau_f = 0.99$ reveal hierarchy of phase space structures.

$v_2$

$v_3$

$v_4$

$v_5$

$v_6$
Identifying ‘ghost rods’: almost-cyclic sets

The zero contour (black) is the boundary between the two almost-invariant sets.

- Three-component AIS made of 3 almost-cyclic sets (ACSs) of period 3
- ACS effectively replace compact region bounded by saddle manifolds
- Also: we see a dynamical remnant of the global ‘stable and unstable manifolds’ of the saddle points, despite no saddle points
Identifying ‘ghost rods’: almost-cyclic sets

Almost-cyclic sets stirring the surrounding fluid like ‘ghost rods’ — works even when periodic orbits are absent!

Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$
Identifying ‘ghost rods’: almost-cyclic sets

Braid of ACSs gives lower bound of entropy via Thurston-Nielsen
— One only needs approximately cyclic blobs of fluid
— Even though the theorems require exactly periodic points!
Topological entropy vs. bifurcation parameter

$h_{TN}$ shown for ACS braid on 3 strands
Movie shows change in eigenvector along branch marked with ‘−□−’ above (a to f), as $\tau_f$ decreases $\Rightarrow$
Bifurcation of ACSs

For example, braid on 13 strands for $\tau_f = 0.92$

Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Thurson-Nielsen for this braid provides lower bound on topological entropy
For various braids of ACSs, the calculated entropy is given, bounding from below the true topological entropy over the range where the braid exists
Chaotic transport: aperiodic, finite-time setting

- Data-driven, finite-time, aperiodic setting
  - e.g., non-autonomous ODEs for fluid flow
- How do we get at transport?
- Recall the flow, \( x \mapsto \phi_{t+T}^t(x) \), where \( \phi : \mathbb{R}^n \to \mathbb{R}^n \)
Identify regions of high sensitivity of initial conditions

- Small initial perturbations $\delta x(t)$ grow like

$$ \delta x(t + T) = \phi_{t+T}^t(x + \delta x(t)) - \phi_{t+T}^t(x) $$

$$ = \frac{d\phi_{t+T}^t(x)}{dx} \delta x(t) + O(||\delta x(t)||^2) $$
Identify regions of high sensitivity of initial conditions

- Small initial perturbations $\delta x(t)$ grow like

$$
\delta x(t + T) = \phi^{t+T}_t(x + \delta x(t)) - \phi^{t+T}_t(x) = d\phi^{t+T}_t(x) \frac{dx}{dt} \delta x(t) + O(\|\delta x(t)\|^2)
$$
Invariant manifold analogs: FTLE-LCS approach

- The finite-time Lyapunov exponent (FTLE),
  \[ \sigma_t^T(x) = \frac{1}{|T|} \log \left\| \frac{d\phi_t^{t+T}(x)}{dx} \right\| \]
  measures the maximum stretching rate over the interval \( T \) of trajectories starting near the point \( x \) at time \( t \)

- Ridges of \( \sigma_t^T \) are candidate hyperbolic codim-1 surfaces; finite-time analogs of stable/unstable manifolds; ‘Lagrangian coherent structures’

\[ (a) \sigma = 3x^4 - 4x^3 - 12x^2 + 18 \frac{12}{(1 + 4y^2)}. \quad (b) \text{Side view.} \quad (c) \text{Curvature measures evaluated along the x-axis (i.e., } y = 0). \quad (d) \text{Close-up.} \]

Fig. 1. Comparison between ridge definitions. Notice that the second-derivative ridge is slightly shorter than the curvature ridge. Therefore we can expect the difference between the two measures to be identically zero or non-existent for all practical purposes. For autonomous systems, \( \sigma \) is constant along a ridge (asymptotically), hence the two definitions of ridge are always identical for such systems.

2.5. Lagrangian coherent structures

Given the graph of a function, the Hessian only represents the curvature of the graph at local extrema, therefore defining a ridge in terms of principal curvatures gives a better physical interpretation and is more intrinsic. However, the notion of a second-derivative ridge is somewhat simpler and more convenient, as we shall see later in this work. Also, we have shown that a second-derivative ridge is always a subset of a principal curvature ridge, and moreover the two definitions are nearly identical for all practical purposes. In addition, the second-derivative definition facilitates computational implementation. Therefore, we define LCS as follows:

\[ \text{cf. Bowman, 1999; Haller \& Yuan, 2000; Haller, 2001; Shadden, Lekien, Marsden, 2005} \]
Invariant manifold analogs: FTLE-LCS approach

Autonomous double-gyre flow
Invariant manifold analogs: FTLE-LCS approach
Invariant manifold analogs: FTLE-LCS approach

Invariant manifolds

LCS

Time-periodic oscillating vortex pair flow
Invariant manifold analogs: FTLE-LCS approach

- We can define the FTLE for Riemannian manifolds\(^3\)

\[
\sigma^T_t (x) = \frac{1}{|T|} \ln \| D\phi^{t+T} \| = \frac{1}{|T|} \log \left( \max_{y \neq 0} \frac{\| D\phi^{t+T} (y) \|}{\| y \|} \right)
\]

with \(y\) a small perturbation in the tangent space at \(x\).

\(^3\)Lekien & Ross [2010] Chaos
Transport barriers on Riemannian manifolds

- Ridges correspond to dynamical barriers\(^3\) or Lagrangian coherent structures (LCS): repelling surfaces for \(T > 0\), attracting for \(T < 0\)

\[\text{cylinder} \quad \text{Moebius strip}\]
Each frame has a different initial time \(t\)

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\(^3\text{Lekien & Ross [2010] Chaos}\)
Atmospheric flows: Antarctic polar vortex
Atmospheric flows: Antarctic polar vortex

ozone data + LCSs (red = repelling, blue = attracting)
Atmospheric flows: Antarctic polar vortex

air masses on either side of a repelling LCS
Atmospheric flows: continental U.S.

LCSs: orange = repelling, blue = attracting
Atmospheric flows and lobe dynamics

orange = repelling LCSs, blue = attracting LCSs

Andrea, first storm of 2007 hurricane season

Atmospheric flows and lobe dynamics

Andrea at one snapshot; LCS shown (orange = repelling, blue = attracting)
Atmospheric flows and lobe dynamics

orange = repelling (stable manifold), blue = attracting (unstable manifold)
Atmospheric flows and lobe dynamics

orange = repelling (stable manifold), blue = attracting (unstable manifold)
Atmospheric flows and lobe dynamics

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out
Atmospheric flows and lobe dynamics

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out
Atmospheric flows and lobe dynamics

Sets behave as lobe dynamics dictates
Atmospheric transport network relevant for aeroecology

Skeleton of large-scale horizontal transport

relevant for large-scale spatiotemporal patterns of important biota e.g., plant pathogens

orange = repelling LCSs, blue = attracting LCSs
2D curtain-like structures bounding air masses
2D curtain-like structures bounding air masses
Pathogen transport: filament bounded by LCS

12:00 UTC 1 May 2007  15:00 UTC 1 May 2007  18:00 UTC 1 May 2007
Pathogen transport: filament bounded by LCS

100 km

(a) (b) (c)

12:00 UTC 1 May 2007 15:00 UTC 1 May 2007 18:00 UTC 1 May 2007

(d) (e) (f)
Coherent sets and set-based definition of FTLE

- Consider, e.g., a flow $\phi_{t+T}^t$ in $(x_1, x_2) \in \mathbb{R}^2$.
- Treat the evolution of set $B \subset \mathbb{R}^2$ as evolution of two random variables $X_1$ and $X_2$ defined by probability density function $f(x_1, x_2)$, initially uniform on $B$, $f = \frac{1}{\mu(B)} \chi_B$, with $\chi_B$ the characteristic function of $B$.
- Under the action of the flow $\phi_{t+T}^t$, $f$ is mapped to $P f$ where $P$ is the associated Perron-Frobenius operator.
- Let $I(f)$ be the covariance of $f$ and $I(P f)$ the covariance of $P f$.

Deformation of a disk under the flow during $[t, t + T]$
Coherent sets and set-based definition of FTLE

- **Definition.** The covariance-based FTLE of $B$ is

\[
\sigma_I(B, t, T) = \frac{1}{|T|} \log \left( \frac{\sqrt{\lambda_{\text{max}}(I(Pf))}}{\sqrt{\lambda_{\text{max}}(I(f))}} \right).
\]

- Reduces to usual definition of FTLE in the limit that the linearization approximation (i.e., line-stretching method) is valid.

Deformation of a disk under the flow during $[t, t + T]$
Coherent sets and set-based definition of FTLE

- The coherence of a set $B$ during $[t, t + T]$ is $\sigma_I(B, t, T)$.
- A set $B$ is almost-coherent during $[t, t + T]$ if $\sigma_I(B, t, T) \approx 0$.

- Captures the essential feature of a coherent set: it does not mix or spread significantly in the domain.
- This definition also can identify non-mixing translating sets.
Coherent sets and set-based definition of FTLE

- The **coherence** of a set $B$ during $[t, t + T]$ is $\sigma_I(B, t, T)$.
- A set $B$ is **almost-coherent** during $[t, t + T]$ if $\sigma_I(B, t, T) \approx 0$.

- Captures the essential feature of a coherent set: it does not mix or spread significantly in the domain.
- This definition also can identify non-mixing **translating** sets.

- **Values of** $\sigma_I(B, t, T)$ **determine the family of sets of various degrees of coherence**.
- Need to set a heuristic threshold on the value of $\sigma_I(B, t, T)$ to determine coherent sets.

- Notice, coherent sets will be separated by ridges of high FTLE, i.e., LCS
FTLE from line-stretching (conventional) during $[0, \tau_f]$
Coherent sets in lid-driven cavity flow

FTLE from covariance-based approach during $[0, \tau_f]$
Coherent sets in lid-driven cavity flow

Sets of coherences $\sigma_I(0, \tau_f) < 0.06$
Coherent sets in lid-driven cavity flow

Compare coherent set with AIS from second eigenvector of $P$
Coherent sets in lid-driven cavity flow

Compare coherent sets with non-coherent set (gray)
Coherent sets in lid-driven cavity flow
Coherent sets in the atmosphere
Coherent sets in the atmosphere

- FTLE from covariance during 24 hours starting 09:00 1 May 2007
Coherent sets in the atmosphere

- Coherent sets during 24 hours starting 09:00 1 May 2007
Optimal navigation in an aperiodic setting?

- Selectively 'jumping' between coherent air masses using control
- Moving between mobile subregions of different finite-time itineraries
Optimal navigation in an aperiodic setting?

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Chaotic transport in higher dimensional systems

- e.g., Hamiltonian systems with multiple potential wells.
- What structures guide transport between potential wells?
  — e.g., restricted three-body problem

\[
H = \frac{1}{2}((p_x + y)^2 + (p_y - x)^2) + \bar{U}(x, y),
\]

where

\[
\bar{U}(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}
\]
Motion in energy surface

- **Energy surface** of energy $E$ is codim-1 surface
  \[ \mathcal{M}(E) = \{ (q, p) \mid H(q, p) = E \} \].

- e.g., in 2 d.o.f., 3D surfaces foliating 4D phase space
Realms of possible motion

- $\mathcal{M}(E)$ partitioned into three realms
  - e.g., Earth realm = phase space around Earth
- Energy $E$ determines their connectivity
Realms of possible motion

Case 1 : $E < E_1$

Case 2 : $E_1 < E < E_2$

Case 3 : $E_2 < E < E_3$

Case 4 : $E_3 < E < E_4$

Case 5 : $E > E_4$
Motion near saddles

Near rank 1 saddles in $N$ degree of freedom system, linearized vector field eigenvalues are

$$\pm \lambda \text{ and } \pm i\omega_j, \ j = 2, \ldots, N$$

Under local change of coordinates

$$H(q, p) = \lambda q_1 p_1 + \sum_{i=2}^{N} \frac{\omega_i}{2} (p_i^2 + q_i^2)$$

to lowest order
Motion near saddles

Equilibrium point is of type saddle $\times$ center $\times \cdots \times$ center ($N - 1$ centers)

the $N$ canonical planes
Motion near saddles

For energy $\hbar$ just above saddle pt, $(q_1, p_1) = (0, 0)$ is normally hyperbolic invariant manifold of bound orbits

$$\mathcal{M}_h = \sum_{i=2}^{N} \frac{\omega_i}{2} (p_i^2 + q_i^2) = h > 0.$$


Motion near saddles

Note that \( M_h \cong S^{2N-3} \)

- \( N = 2 \), the circle \( S^1 \), a single periodic orbit
- \( N = 3 \), the 3-sphere \( S^3 \), a set of periodic and quasi-periodic orbits

\[ N = 2 \]
\[ N = 3 \]

the \( N \) canonical planes
Motion near saddles

Note that $\mathcal{M}_h \simeq S^{2N-3}$

- $N = 2$, the circle $S^1$, a single periodic orbit
- $N = 3$, the 3-sphere $S^3$, a set of periodic and quasi-periodic orbits

Four “cylinders” or tubes of asymptotic orbits: stable, unstable manifolds, $W^s_\pm(\mathcal{M}_h), W^u_\pm(\mathcal{M}_h), \simeq S^1 \times \mathbb{R}$ for $N = 2$
Motion near saddles: 2 d.o.f.

- **B**: bounded orbits (periodic/quasi-periodic): $S^1$
- **A**: asymptotic orbits to 1-sphere: $S^1 \times \mathbb{R}$ (tubes)
- **T**: transit and **NT**: non-transit orbits.
Tube dynamics: All motion between adjacent realms connected by necks around saddles must occur through the interior of tubes\textsuperscript{6}

\textsuperscript{6}Koon, Lo, Marsden, Ross [2000,2001,2002], Gómez, Koon, Lo, Marsden, Masdemont, Ross [2004]
Related systems

- Much work in celestial mechanics
- Results apply to problems in chemistry, biomechanics, ship capsize
Tubes leading to capsize

- Ship motion is Hamiltonian,

\[ H = p_x^2/2 + R^2 p_y^2/4 + V(x, y), \]

\[ V(x, y) \]
Tubes leading to capsize

Poincaré section
transition state
Tubes leading to capsize

- Wedge of trajectories leading to imminent capsize

- Tubes are a useful paradigm for predicting capsize even in the presence of random forcing, e.g., random waves

- Could inform control schemes to avoid capsize in rough seas
Final words on chaotic transport

What are robust descriptions of transport which work in data-driven aperiodic, finite-time settings?

- Possibilities: finite-time lobe dynamics / symbolic dynamics may work — finite-time analogs of homoclinic and heteroclinic tangles

- Probabilistic, geometric, and topological methods — invariant sets, almost-invariant sets, almost-cyclic sets, coherent sets, stable and unstable manifolds, Thurston-Nielsen classification, FTLE, LCS

- Many links between these notions — e.g., LCS locate analogs of stable and unstable manifolds — boundaries between coherent sets are naturally LCS — periodic points \(\Rightarrow\) almost-cyclic sets — their ‘stable/unstable invariant manifolds’ \(\Rightarrow\) ???
Main Papers:


