Geometric and probabilistic descriptions of chaotic phase space transport: stirring by braiding of almost-cyclic sets

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Motivation: complex fluid motion, mixing, and control
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Atmosphere over North America. Lagrangian coherent boundaries: orange = repelling, blue = attracting
Motivation: complex fluid motion, mixing, and control

Table top fluid experiment. Lagrangian coherent boundaries: red = repelling, blue = attracting
Motivation: complex fluid motion, mixing, and control

- Selectively 'jumping' between coherent sets using control
- Moving between mobile subregions of different finite-time itineraries
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green=uncontrolled, red=controlled
Stirring fluids with solid rods

- Turbulent mixing: spoon in coffee
- Laminar mixing: 3 ‘braiding’ rods in glycerin
Topological chaos through braiding of stirrers

Topological chaos is ‘built in’ the flow due to the topology of boundary motions.

$R_N$: 2D fluid region with $N$ stirring ‘rods’

- stirrers move on periodic orbits
- stirrers = solid objects or fluid particles
- stirrer motions generate diffeomorphism $f: R_N \rightarrow R_N$
- stirrer trajectories generate braids in 2+1 dimensional space-time
Thurston-Nielsen classification theorem

- A stirrer motion $f$ is isotopic to a stirrer motion $g$ of one of three types (i) finite order (f.o.): the $n$th iterate of $g$ is the identity (ii) pseudo-Anosov (pA): $g$ has Markov partition with transition matrix $A$, topological entropy $h_{TN}(g) = \log(\lambda_{PF}(A))$, where $\lambda_{PF}(A) > 1$ (iii) reducible: $g$ contains both f.o. and pA regions

- $h_{TN}$ computed from ‘braid word’, e.g., $\sigma_1^{-1}\sigma_2$

- $\log(\lambda_{PF}(A))$ provides a lower bound on the true topological entropy
Topological chaos in a viscous fluid experiment

Move 3 rods on ‘figure-8’ paths through glycerin

- stirrers move on periodic orbits in two steps
- Thurston-Nielsen theorem gives a lower bound on stretching:
  \[ \lambda_{TN} = \frac{1}{2} (3 + \sqrt{5}) \]
  \[ h_{TN} = \log(\lambda_{TN}) = 0.962 \ldots \]

non-trivial material lines grow like
\[ l \sim l_0 \lambda^n \]
\[ \lambda \geq \lambda_{TN} \]
Topological chaos in a viscous fluid experiment

finite order

pseudoAnosov

\( R_+ \)

\( L_+ \)

\( \sigma_1 \)

\( \sigma_2 \)

\( \sigma_1^{-1} \)

\( \sigma_2 \)
‘Stirring’ with fluid particles

point vortices in a periodic domain

one rod moving on an epicyclic trajectory

Fluid is wrapped around ‘ghost rods’ in the fluid
– flow structure assists in the stirring
Ghost rods in microfluidics mixer

- Lid-driven cavity flow, periodic vector field

- System has parameter $\tau_f$, which we treat as a bifurcation parameter — critical point $\tau_f^* = 1$

- $t \in [n\tau_f, (n+1)\tau_f/2)$, right two points exchange clockwise
- $t \in [(n+1)\tau_f/2, (n+1)\tau_f)$, left two points exchange counter-clockwise

streamlines for $\tau_f = 1$  
tracer blob ($\tau_f > 1$)
Stirring protocol $\Rightarrow$ braid $\Rightarrow$ topological entropy

- Consider period-$\tau_f$ map
- For $\tau_f = 1$, period 3 points act as ‘ghost rods’
- Their braid $\Rightarrow h_{TN} = 0.96242$ from TNCT
- Actual $h_{\text{flow}} \approx 0.964$ obtained numerically
- $\Rightarrow h_{TN}$ is an excellent lower bound
Identifying ‘ghost rods’: periodic points

period-$\tau_f$ map for $\tau_f$ just above 1

- At $\tau_f = 1$, parabolic period 3 points of map
- $\tau_f > 1$, **elliptic / saddle points** of period 3 — streamlines around groups resemble fluid motion around a solid rod $\Rightarrow$
- $\tau_f < 1$, **periodic points vanish**
Consider $\tau_f < 1$
Identifying ‘ghost rods’?

period-$\tau_f$ map for $\tau_f < 1 \Rightarrow$ no ‘obvious’ structure

- Note the absence of any elliptical islands
- No periodic orbits of low period were found
- In practice, even when such low-order periodic orbits exist, they can be difficult to identify
- But phase space is not featureless
Almost-cyclic set approach

- Identify almost-invariant sets (AISs, as discussed in previous talks)
- Relatedly, almost-cyclic sets (ACSs) (Dellnitz & Junge [1999])
- Create box partition of phase space \( \mathcal{B} = \{B_1, \ldots B_q\} \), with \( q \) large
- Consider a \( q \)-by-\( q \) Ulam-Galerkin matrix, \( P \), where

\[
P_{ij} = \frac{m(B_i \cap f^{-1}(B_j))}{m(B_i)},
\]

the transition probability from \( B_i \) to \( B_j \) using, e.g., \( f = \phi_t^T \), computed numerically

- Identify AISs and ACS via spectrum of \( P \)
Identifying ‘ghost rods’: almost-cyclic sets

- For $\tau_f > 1$ case, where periodic points and manifolds exist...
- Agreement between ACS boundaries and manifolds of periodic points
- Known previously\(^1\) and applies to more general objects than periodic points, i.e. normally hyperbolic invariant manifolds (NHIMs)

Identifying ‘ghost rods’: almost-cyclic sets

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Identifying ‘ghost rods’: almost-cyclic sets

period-$\tau_f$ map for $\tau_f < 1 \Rightarrow$ no ‘obvious’ structure

- Return to $\tau_f < 1$ case, where no periodic orbits of low period known
- What are the AISs and ACSs here?
- Consider $P_{t}^{t+\tau_f}$ induced by family of period-$\tau_f$ maps $\phi_{t}^{t+\tau_f}$, $t \in [0, \tau_f)$
Identifying ‘ghost rods’: almost-cyclic sets

Top eigenvectors for $\tau_f = 0.99$ reveal hierarchy of phase space structures

$v_2$

$v_3$

$v_4$

$v_5$

$v_6$
Identifying ‘ghost rods’: almost-cyclic sets

The zero contour (black) is the boundary between the two almost-invariant sets.

- Three-component AIS made of 3 ACSs of period 3
- ACS effectively replace periodic orbits for TNCT
Identifying ‘ghost rods’: almost-cyclic sets

Almost-cyclic sets stirring the surrounding fluid like ‘ghost rods’ — works even when periodic orbits are absent!

Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$
Identifying ‘ghost rods’: almost-cyclic sets

Braid of ACSs gives lower bound of entropy via Thurston-Nielsen
— One only needs approximately cyclic blobs of phase space
— But, theorems apply only to periodic points
Topological entropy vs. bifurcation parameter

$h_{TN}$ shown for ACS braid on 3 strands
Consider change in eigenvector $z$ along continuous branch marked with ‘−□−’ above (from a to f), as $\tau_f$ decreases $\Rightarrow$

$z$Inspired by Junge, Marsden, Mezic [2004]
Bifurcation of ACSs — braid on 13 strands

For example, braid on 13 strands for $\tau_f = 0.92$

Movie shown is second eigenvector for $P_t^{t+\tau_f}$ for $t \in [0, \tau_f)$

Thurson-Nielsen for this braid provides lower bound on topological entropy
Bifurcation of ACSs — braid on 13 strands

(a) Initial state

(b) First half-period

(c) Second half-period

(d) State after 1 period
Bifurcation of ACSs — braid on 13 strands
For various braids of ACSs, the calculated entropy is given, bounding from below the true topological entropy over the range where the braid exists.
Non-autonomous, non-periodic, finite-time setting

- Data-driven, finite-time, non-periodic setting — e.g., from experimental fluid measurements, observations
- Are there, e.g., braids in realistic fluid flows?

LCSs: orange = repelling, blue = attracting
Atmospheric flows: hurricanes

orange = repelling curves, blue = attracting curves

Andrea, first storm of 2007 hurricane season

Atmospheric flows: hurricanes

Andrea at one snapshot; Lagrangian coherent boundaries shown
Atmospheric flows: lobe dynamics to find braids

orange = repelling (stable manifold), blue = attracting (unstable manifold)
Atmospheric flows: lobe dynamics to find braids

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Atmospheric flows: lobe dynamics to find braids

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out
Atmospheric flows: lobe dynamics to find braids

Portions of lobes colored; magenta = outgoing, green = incoming, purple = stays out
Atmospheric flows: lobe dynamics to find braids

Sets behave as lobe dynamics dictates $\Rightarrow$ form braid, but no periodicity
Atmospheric flows: Antarctic polar vortex
Atmospheric flows: Antarctic polar vortex

ozone data + Lagrangian coherent boundaries (red = repelling, blue = attracting)
Speculation: trends in eigenvalues/vectors for prediction

- Different eigenvectors can correspond to dramatically different behavior.
- Some eigenvectors increase in importance while others decrease.
- Can we predict dramatic changes in system behavior?
- e.g., splitting of the ozone hole in 2002, using only data before split.
Final words

- Almost-cyclic sets enable application of the TNCT even in the absence of low-order periodic orbits.
  - For engineering systems, can design for mixing using ACSs
  - For natural systems, ghost rod/ACS paradigm may aid interpretation

- Connection between finite-time lobe dynamics and braids

- Bifurcation of phase space structure revealed through bifurcation of AIS/ACSs, braid bifurcations, etc.

- Prediction of dramatic changes in system behavior using changing order of eigenvectors?
The End

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