Abstract

In this paper we present a method to determine the initial conditions that define nonlinear normal modes (NNM’s) in conservative nonlinear structural models of the form

\[ [M][\ddot{x}] + [K][x] + \{N(x)\} = \{0\} \]  

(1)

where \{x\} is the n-vector of physical coordinates, \{N(x)\} is a static nonlinear function of the coordinates, and the remaining terms have their usual meanings.

We seek the individual NNM modal manifolds that are tangent to the modal eigenspaces of the associated linear system at the origin of the state space. The motions sought on these manifolds are periodic. To fix ideas, consider a 40 DOF spring/mass oscillator chain for which the associated linear system has 40 unit masses and 40 linear springs of unit stiffness, with the left hand mass \( m_1 \) grounded and the right hand mass \( m_{40} \) free. The linear natural frequencies are, in radians/second, \( \omega_i = (0.0388, 0.1163, 0.1936, 0.2707, 0.3473, 0.4234, \ldots) \). A single nonlinear spring of Duffing type is connected to the right hand mass \( m_{40} \) and to ground, producing a force \(-\varepsilon x_{40}^3\) on \( m_{40} \). This system exhibits numerous internal resonance conditions.

We seek a motion on the 1st NNM of the nonlinear system for the case \( \varepsilon = 1.0 \) by starting with the (zero velocity) initial condition \{x(0)\} = \{\phi_1\}, the normalized 1st mode shape of the associated linear system. We then integrate equations (1) numerically to generate a solution. Because the initial condition is not on the 1st modal manifold, the motions are not periodic. The solutions for several of the coordinates \( x_1, x_{10}, \) and \( x_{40} \) are shown in Figure 1.

In order to find the NNM that is “not too far” from the 1st linear mode shape, we note that the exact NNM may be defined by the initial conditions \{x(0)\} = [\Phi]\{y(0)\}, where [\Phi] is the modal matrix of the associated linear system and where the initial values \( y_i(0) \) of the modal coordinates \( y_i \) are chosen correctly (with \( y_i(0) = 1 \)). The linear modal coordinates \( y_i \) are governed by the equations of motion.
\[
\ddot{y}_i + \omega_i^2 y_i = -\varepsilon \phi_{i,40} x_{40}^3; \quad \text{where } x_{40} = \sum_j \phi_{40,j} y_j \tag{2}
\]

Several of the linear modal coordinate histories associated with the solutions of Figure 1 are shown in Figure 2.

The following are revealed from the solutions for the modal coordinates: 1) the first modal coordinate \( y_1 \) behaves essentially as a single degree of freedom Duffing oscillator for which the hardening nonlinear effect has shifted the oscillation frequency to a value of 0.0505 rad/sec (from the linear value of 0.0388); this frequency shift has the effect of essentially eliminating internal resonance effects in the response; 2) essentially, the modal coordinates \( y_2 \) through \( y_{40} \) are excited harmonically at the frequencies 0.0505 and 0.1515 arising in the \( x_{40}^3 \) terms on the right hand sides of equations (2). Thus, the modal coordinate responses \( y_i \) for coordinates \( y_2 \) through \( y_{40} \) consist of “steady state” forced responses at one and three times the fundamental “exciting frequency” of 0.0505 rad/sec, along with the “transient” response at the natural frequency \( \omega_i \) (most easily visible in \( y_6 \) in Figure 2).

The initial displacement \( \{x(0)\} \) needed to place the state in the 1st modal manifold is determined by selecting the nonzero initial values of \( y_2 \) through \( y_{40} \) that result in elimination of the transient response in each of these modal coordinates, so that each modal coordinate responds periodically with fundamental frequency of 0.0505 rad/sec. This ensures that the physical coordinate response is also periodic, corresponding to the 1st NNM. The method for determination of the appropriate initial values of the modal coordinates \( y_i \) will be discussed at the conference. For the problem considered here, the resulting (periodic) modal coordinate histories are shown in Figure 3. The associated physical coordinates \( x_i \) are also periodic.

Figure 2 – Four of the linear modal coordinates for the solution of Figure 1: \( y_1 \) (top), \( y_2 \) (middle solid), \( y_3 \) (middle dashed), and \( y_6 \) (bottom).

Figure 3 – Same linear modal coordinates as in Figure 2, but simulation was done for nonzero initial values of \( y_2 \) through \( y_{40} \) selected to eliminate transient response in these modal coordinates.