Obtaining the dynamic equations, their simulation, and animation for \( n \) pendulums using Maple

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1 Introduction
Dynamic systems modeling and simulation of multiple pendulums have been of great academic interest due to their nonlinearity and coupling. For our work we tried to develop a general form for the modeling and simulation of multiple pendulums just limited by the memory and CPU clock of the computer being used.

Lagrangian dynamics can be used for systems with \( n \) design variables. Any of the several formulations of the fundamental dynamic laws could be used to derive the Lagrange equations.

2 Modeling
If \( n \) is the number of design variables or degrees of freedom (dof) of the system for a conservative system [8] and no external forces are applied, we have:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \ldots, n
\]

(1)

where
\[
L = T - V, \quad \text{Lagrange equation for conservative systems},
\]
\[
T = \text{Kinetic energy of the system},
\]
\[
V = \text{Potential energy of the system},
\]
\[
q_i = \text{\( i \)-th Generalized coordinate (dof)}
\]
\[
\dot{q}_i = \frac{d}{dt} q_i
\]

In our case, the generalized coordinates are angles (counterclockwise is positive) by each pendulum as shown in Figure 1.

We first specify the number of pendulums for the system. Then, we obtain the \((x,y)\) components of the position vectors \( r_i \) for each mass. The origin of the standard reference frame is at the pin of the first pendulum.

\[
r := \text{seq}(\{x[i], y[i], i=1..n\});
\]

Once we have the position vectors, we can find the velocity vectors. For doing so, we take advantage of Maple functionality by creating a lambda function which is a small anonymous function to be applied (mapped)

\[
v2 := \text{map}(\text{velCuad}, \text{vv});
\]

The kinetic energy requires the square of the velocities and to doing so we define a function that will be mapped to the velocities. The function obtains the sum of the square of the components and also combine and factor the elements to show the velocities in compact form:

\[
v2 := \left[ L_1^2 \ddot{q}_1(t)^2, L_1^2 \ddot{q}_1(t)^2 + 2 \cos(q_2(t) - q_1(t)) L_2 L_1 \dot{q}_2(t) \dot{q}_1(t) + L_2^2 \dot{q}_2(t)^2 \right]
\]

Some useful sequences (position, velocities, accelerations, masses, and lengths) are defined to collect common terms in the resulting expressions.
We obtain the kinetic energy for each mass and collect common terms:

\[ e_T := \text{collect}(\text{simplify}(\sum \frac{1}{2} m[i] v_2[i], 'i'=1..n)), \]

\[ [seqM, seqV, seqL]; \]

Now we obtain the potential energy which is the vertical distance measured from its stable equilibrium position:

\[ e_V := \text{collect}(\text{simplify}(\sum (\sum L[j] 'j'=1..i)+y[i]*m[i]*g, 'i'=1..n)), [g, seqL, seqM, seqP]; \]

And form the Lagrange equation:

\[ \text{Lag} := e_T - e_V; \]

The built-in functions to differentiate in Maple can only differentiate with respect to a variable and not a function. Due to this, some substitutions were needed in order to obtain the elements of the Lagrange Equation, i.e., \( \dot{q}_i \) was substituted by \( q_d \) to obtain \( \frac{\partial L}{\partial q_d} \), then counter-substituted to obtain \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \):

\[ \text{EcMov} := \{ \text{seq} \left( \text{diff}(\text{subs}(q_d=\text{diff}(q[i](t),t), \text{diff}(\text{subs}(\text{diff}(q[i](t),t)=q_d, \text{Lag}), q_d)), t), 'i'=1..n) \}; \]

We performed a similar procedure to obtain \( \frac{\partial L}{\partial q_i} \):

\[ \text{EcMov} := \{ \text{seq} [\text{EcMov}[i] - \text{subs}(q[i](t)=q[i], \text{diff}(\text{subs}(q[i]=q[i], \text{Lag}), q[i])), 'i'=1..n) \}; \]

\[ \text{EDMov} := \{ \text{map} (\text{collect}, \text{EcMov}, [\text{seqA}, \text{seqV}, \text{seqM}, \text{seqL}]) \}; \]

3 Simulation

A function was developed to allow random initial positions varying in the range \([-315.15, 315.15]\) radians. The other parameters were fixed.

\[ qRand := \text{rand}(-315..315)/100.0; \]

\[ L := \{ \text{seq}(i=1..n) \}; \]

\[ g := 9.81; \]

\[ L := \{ \text{seq}(i=1..n) \}; \]

\[ m := \{ \text{seq}(i=1..n) \}; \]

\[ q0 := \{ \text{seq}(qRand(), i=1..n) \}; \]

\[ qd0 := \{ \text{seq}(qRand(), i=1..n) \}; \]

The system is coupled; it is convenient to help Maple find the State Space representation [3] in order to apply a numerical method to solve the differential equations. The default numeric method in the Maple function dsolve is a Fehlberg fourth-fifth order Runge-Kutta method [1] with degree four interpolant. However, if the system has more than six pendulums, an in-lin function with a fourth order Runge-Kutta method is used.

Since we require expressions for the higher order derivatives of the generalized coordinates uncoupled, we can form a linear system having them as the unknowns and solve it to uncouple the system on the higher order derivatives:

\[ \text{seqLS} := \{ \text{seq} [\text{seqA}[i]=z[i], 'i'=1..n] \}; \]

\[ A := \text{genmatrix}(\text{subs}(\text{seqLS}, \text{EDMov}, [\text{seq}(z[i], 'i'=1..n)], b)); \]

and solve them to obtain the State Space representation of the system:

\[ zz := \text{linsolve}(A, b); \]

\[ \text{seqCIP} := \{ \text{seq}(z[i]=z[i], 'i'=1..n) \}; \]

\[ \text{seqCIV} := \{ \text{seq}(\text{D}(z[i])(0)=0, 'i'=1..n) \}; \]

Once having this, we can solve them (in this example dsolve was used):

\[ \text{EDSol} := \text{dsolve}(\{ \text{seq} [\text{seqA}[i]=zz[i], 'i'=1..n], \}

> \text{ECIP}, \text{ECIV}, \{ \text{seqP}, \text{numeric} \}); \]

The animation requires fewer points than the simulation generates. Due to this, some filtering is required. Several tests in addition to the two-pendulum system were performed by adding a pendulum at a time. The simulation time increased specially when the number of pendulums exceeds six (up to ten hours for ten pendulums).

4 Conclusions

The Lagrange analysis was presented to obtain the differential equations of motion for an \( n \)-pendulum system with no friction and no external forces using most of the Maple functionality. The symbolic, numeric, and graphic manipulation of Maple provides an excellent platform to derive not just the equations of motion but also to simulate them and generate an animation by just changing the number of pendulums in the system.

References


